

ANALYSIS

Finite-Sample Distortion
Measures: Unified Risk and
Gain via Scenario Weights



Finite-Sample Distortion Measures: Unified Risk and Gain via Scenario Weights

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Version 1.0

February 2026*

Abstract

We study monetary and coherent measures of portfolio risk and gain when returns are represented by a finite scenario set, as in historical or Monte Carlo simulation. In this setting, many quantile-based functionals—including Value at Risk and Conditional Value at Risk—become linear forms in the order statistics and are therefore fully characterized by scenario-weight vectors. This leads to an implementation-friendly class of *distribution measures* that treats downside risk and upside potential within a unified framework. We derive explicit finite-sample weights for VaR/CVaR and their gain counterparts (GaR/CGaR), obtained by applying the risk measure to the short position. We further introduce *equivalent weight vectors* that preserve reported values on a given scenario set while enabling smoother or more robust weight profiles, and we present a matrix formulation for fully vectorized evaluation over large panels of measures. Numerical results for major digital assets illustrate the approach and quantify tail asymmetries between losses and gains.

1 Introduction

Risk management is a central task in asset and portfolio management. Portfolio returns are uncertain, and investors care not only about expected performance but also about the size and likelihood of potential losses. Regulators, clients, and internal risk committees therefore require quantitative measures that summarize downside risk, can be reported in a consistent way, and can be used to set limits, capital buffers, and risk budgets.

In modern practice, these requirements are often addressed in a *scenario-based* setting. Portfolio returns are generated or observed as a large but finite set of outcomes, coming either from historical data (historical simulation) or from numerical methods (Monte Carlo simulation and stress scenarios). Advances in computing hardware and numerical methods now make it routine to work with tens or hundreds of thousands of scenarios, even for portfolios of complex or illiquid instruments. In such cases, risk managers are less constrained to fit simple continuous parametric models (for example, Gaussian or elliptical distributions) and can instead treat the empirical distribution of the simulated or historical sample as the primary input to risk measurement.

This finite-sample viewpoint is particularly relevant for digital assets such as cryptocurrencies, related tokens, and their derivatives. Returns in these markets are typically more volatile than those of traditional asset classes and often exhibit heavy tails and abrupt moves, so tail-sensitive measures and robust implementations are essential. Moreover, digital-asset markets trade continuously across many venues, so exposures can change quickly and must be monitored on large panels of scenarios,

*Latest review February 25, 2026

both for risk control and for comparing downside risk to upside potential; see, for example, [1] for portfolio tail-risk measurement in cryptocurrencies.

Two workhorse measures dominate both regulation and practice: *Value at Risk* (VaR) and *Conditional Value at Risk* (CVaR), the latter also known as *Expected Shortfall*. VaR reports a tail quantile of losses and remains widely used for reporting and limits, but it is not coherent in general: subadditivity may fail for non-elliptical distributions or discrete samples, so VaR does not always reward diversification; see, e.g., [2]. By contrast, CVaR averages losses beyond the VaR threshold and is coherent under mild conditions; see [3], and [4] for an optimization representation that turns CVaR computation and minimization into a convex program. More broadly, [5] place VaR, CVaR, and related functionals in the general theory of monetary and convex risk measures, providing dual representations and stability properties; see also the Basel market-risk standard for the regulatory role of Expected Shortfall [6].

The theoretical literature typically formulates these concepts in terms of abstract random variables on a probability space, with results expressed in distributional or functional-analytic form (see e.g. [7]). In practice, however, risk is often computed directly on a finite ordered sample of returns. This shift is not merely notational: finite-sample definitions can be sensitive to the addition or removal of a small number of scenarios (see e.g. [8]).

In this paper we take the finite, ordered scenario sample as the primitive input and by representing each measure through a vector of scenario weights applied to the order statistics. This makes the simulation-based setting explicit: axioms such as cash-additivity and monotonicity translate into simple algebraic constraints on the weights, and computation reduces to a dot product (or, for many measures at once, a matrix multiplication). The resulting framework is closely aligned with the distortion and spectral viewpoints—risk and performance functionals constructed as weighted averages of quantiles—which are standard in the continuous setting [9].

We adopt the sign convention that positive values denote gains and negative values denote losses, and we work with cash returns in a fixed currency (the same formulas apply to percentage returns).

Our contributions are as follows:

- We provide a concise axiomatic review of monetary and coherent risk measures and position VaR and CVaR within this framework.
- We introduce *gain measures* by applying risk measures to short positions and show how the main axioms translate transparently to the upside.
- We define *distribution measures* as a unified class encompassing both risk and gain measures, distinguished by a measure sign and normalization of the associated weights.
- In the finite-sample setting, we focus on *distortion measures*, i.e. distribution measures that can be written as linear combinations of the ordered sample with weights depending only on the confidence level and the sample size. We derive explicit weight-vector representations for VaR/CVaR and their upside counterparts GaR/CGaR.
- We introduce *equivalent weight vectors*—distinct weights that agree numerically on a given scenario set but can exhibit improved qualitative properties—and we present an implementation-ready matrix formulation for computing large panels of measures in a fully vectorized way.
- For completeness, appendix A connects the finite-sample weight-vector representation to the general distributional definition of distortion measures.

Taken together, these results provide a finite-sample “calculus” that links the abstract axioms of risk measurement to the scenario-based workflows used in practice, and that treats downside risk and upside potential within a single, implementation-friendly framework.

2 Risk Measures

Suppose that we have a portfolio composed of a number of assets and we want to quantify its risk. The results in this section apply both to cash returns and to percentage returns; however, to keep notation and interpretation simple, we assume throughout that portfolio returns are measured as cash in a given currency.

It is convenient to model the (one-period) portfolio return as a random variable, so that we can use standard tools from probability theory. We therefore denote by X the random variable representing the future portfolio cash return, with positive values corresponding to gains and negative values to losses.

In the main body of the paper we use probabilistic terminology informally and do not fix a specific underlying probability space. For a fully formal treatment of risk measures (including dual representations) see, for example, references [5] and [10]. In appendix A we give a distributional definition of distortion measures and derive a general result linking risk and gain measures.

2.1 Monetary risk measures

A *risk measure* assigns to each portfolio return X a real number $\rho(X)$ intended to summarize its downside risk according to a given criterion. In practice, multiple risk measures are often computed for the same portfolio so that risk is described from different perspectives.

Let \mathcal{L} be a linear space of random variables modeling portfolio returns. A *monetary risk measure* is a map $\rho : \mathcal{L} \rightarrow \mathbb{R}$ that satisfies the following axioms:

- **Normalization:** $\rho(0) = 0$.
- **Monotonicity:** if $X_1 \leq X_2$ almost surely, then $\rho(X_2) \leq \rho(X_1)$.
- **Cash-additivity (cash-invariance):** for all $c \in \mathbb{R}$, $\rho(X + c) = \rho(X) - c$.

Normalization means that a position with identically zero return has zero risk.

Monotonicity encodes the idea that a portfolio with uniformly larger payoffs cannot be riskier: if X_2 dominates X_1 scenario by scenario, then its risk should be no larger.

Cash-additivity fixes the interpretation of the risk measure as a *cash capital requirement*. Adding a sure amount c to the portfolio shifts every outcome upward by c and therefore reduces required capital by exactly c . Equivalently, one can interpret $\rho(X)$ as the amount of certain cash that must be added to X to make the position acceptable under the chosen risk criterion.

The axiomatic definition is useful because it separates economically meaningful properties from arbitrary loss summaries: many functionals can be computed from data, but only those satisfying the axioms behave consistently under dominance and cash shifts.

2.2 Coherent risk measures

The axioms above are a minimal baseline, but they do not encode how risk should behave under scaling or diversification. This motivates the notion of *coherent* risk measures (see e.g. reference [2]).

A risk measure is *coherent* if it satisfies, in addition to the monetary axioms, the following two axioms:

- **Positive homogeneity:** $\rho(\lambda X) = \lambda \rho(X)$ for all $\lambda > 0$.
- **Subadditivity:** $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$.

Positive homogeneity means that scaling the position scales the risk by the same factor, which is natural when risk grows proportionally with exposure. (As noted, this axiom may be inappropriate in settings where liquidity effects or market impact make risk superlinear in position size.)

Subadditivity formalizes the diversification principle: combining portfolios should not increase risk beyond the sum of stand-alone risks. A coherent risk measure therefore rewards diversification and behaves consistently with respect to aggregation.

Note that positive homogeneity and subadditivity imply convexity of the risk measure:

$$\rho(\lambda X_1 + (1 - \lambda) X_2) \leq \lambda \rho(X_1) + (1 - \lambda) \rho(X_2),$$

for all $\lambda \in [0, 1]$ and all $X_1, X_2 \in \mathcal{L}$.

Coherent risk measures also admit powerful representation results (for example, as worst-case expected losses under suitable sets of probability measures), which underpin both theoretical analysis and numerical methods; see, e.g., references [2] and [5].

Examples: VaR and CVaR. Two canonical examples are Value at Risk (VaR) and Conditional Value at Risk (CVaR, or Expected Shortfall). VaR is defined as the smallest loss such that the probability of observing a loss greater than or equal to this value is at most $1 - \ell$. *Conditional Value at Risk* (Expected Shortfall) at level ℓ is defined as the average loss beyond the VaR threshold. In later sections we specialize these definitions to the finite-sample, scenario-based setting and show that both VaR and CVaR admit simple weight-vector representations as linear functionals of ordered samples.

3 Gain Measures and Distribution Measures

Risk measures quantify the downside risk of a portfolio, that is, its potential losses. However, a portfolio manager or an investor is usually also interested in the upside of a portfolio. We introduce the concept of *gain measures*, which quantify the upside potential of a portfolio, that is, its potential profits.

Gain measures inherit the main axioms of risk measures under a simple sign transformation, and can therefore be treated within the same framework. Therefore we later introduce the concept of *distribution measures*, which unifies the two concepts of risk and gain measures.

3.1 Gain measures

In principle, one could define gain measures axiomatically, using postulates similar to those for risk measures. However, it is more intuitive and convenient to define gain measures starting from risk measures applied to short positions.

Let X denote the random variable representing the future cash return of a portfolio (positive for gains, negative for losses). Consider the portfolio that is exactly short this position, whose return is $-X$. The gain of the original portfolio corresponds to the loss of the short portfolio. This suggests the following definition. Given a risk measure ρ , we define the associated *gain measure* γ_ρ as the risk of the short position, i.e.:

$$\gamma_\rho(X) = \rho(-X), \tag{1}$$

where, again, $-X$ denotes the negative of the random variable X , so that gains are turned into losses and losses into gains.

Note that in reference [10] the authors consider $\rho(-X)$ as a way to define the properties of risk measures, without explicitly defining gain measures. We believe that gain measures are useful in their own right and merit an explicit treatment alongside risk measures.

We now discuss how the axioms of risk measures translate to the associated gain measures defined by equation (1).

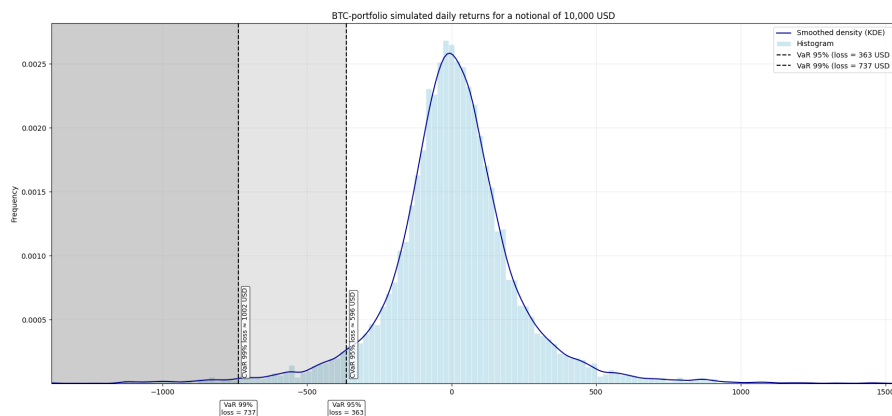


Figure 1: VaR and CVaR for a portfolio fully invested in Bitcoin (BTC)

Normalization If the risk measure ρ is normalized, then the gain measure γ_ρ is also normalized. Indeed, we have

$$\gamma_\rho(0) = \rho(-0) = \rho(0) = 0,$$

where 0 is the zero random variable. Thus, a portfolio with zero returns has a zero gain, as expected.

Monotonicity The monotonicity property reverses its direction when we move from risk measures to gain measures. Let ρ be a monotone risk measure and let X_1 and X_2 be two portfolios such that $X_1 \leq X_2$ almost surely. Then

$$-X_2 \leq -X_1.$$

By monotonicity of ρ we obtain

$$\rho(-X_1) \leq \rho(-X_2),$$

which, using the definition of γ_ρ , can be written as

$$\gamma_\rho(X_1) = \rho(-X_1) \leq \rho(-X_2) = \gamma_\rho(X_2).$$

Therefore, if one portfolio has returns that are always less than or equal to those of another portfolio, its gain (according to γ_ρ) cannot be larger. In other words, we expect the gain of a portfolio with higher returns to be larger.

Cash-additivity The cash-additivity property of risk measures translates directly to gain measures, with a natural change of sign.

Let ρ be a cash-additive risk measure, so that for any portfolio Y and any constant c we have $\rho(Y + c) = \rho(Y) - c$. Then, for any portfolio X and cash amount c , we obtain

$$\gamma_\rho(X + c) = \rho(-(X + c)) = \rho(-X - c) = \rho(-X) + c = \gamma_\rho(X) + c.$$

Thus, the gain measure γ_ρ is cash-additive with the cash amount added, rather than subtracted. Adding a sure amount of cash c to the portfolio increases its gain measure by c .

Positive homogeneity Assume that the risk measure ρ is positively homogeneous, that is, $\rho(\lambda X) = \lambda \rho(X)$ for all $\lambda > 0$. Then the associated gain measure γ_ρ is also positively homogeneous. Indeed, for any $\lambda > 0$ we have

$$\gamma_\rho(\lambda X) = \rho(-\lambda X) = \rho(\lambda(-X)) = \lambda \rho(-X) = \lambda \gamma_\rho(X).$$

So scaling the portfolio by a positive factor λ scales the gain measure by the same factor.

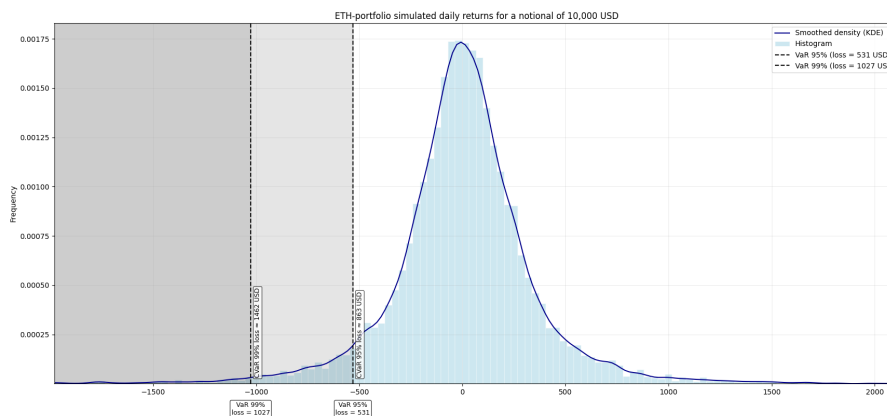


Figure 2: VaR and CVaR for a portfolio fully invested in Ethereum (ETH)

Subadditivity The subadditivity property carries over from risk measures to gain measures. Let ρ be a subadditive risk measure and let X_1 and X_2 be two portfolio returns. Then

$$\rho(-(X_1 + X_2)) = \rho(-X_1 - X_2) \leq \rho(-X_1) + \rho(-X_2),$$

where we used subadditivity of ρ . Using the definition of γ_ρ , this becomes

$$\gamma_\rho(X_1 + X_2) \leq \gamma_\rho(X_1) + \gamma_\rho(X_2).$$

Therefore, the gain measure γ_ρ is subadditive: the gain of the sum of two portfolios is not larger than the sum of their individual gains, according to this measure.

3.2 Distribution measures

We can place both risk measures and gain measures in a single framework of *distribution measures*. In other words, a distribution measure is a functional $\mathcal{D} : \mathcal{L} \rightarrow \mathbb{R}$ and a measure sign $m_s \in \{+1, -1\}$, with $m_s = +1$ for gain and $m_s = -1$ for risk, such that we have the following properties:

1. *Normalization*: $\mathcal{D}(0) = 0$.
2. *Monotonicity*: If $X_1 \leq X_2$ almost surely, then $m_s \cdot \mathcal{D}(X_1) \leq m_s \cdot \mathcal{D}(X_2)$.
3. *Cash-additivity*: $\mathcal{D}(X + c) = \mathcal{D}(X) + m_s \cdot c$ for any constant c .
4. *Positive homogeneity*: $\mathcal{D}(\lambda \cdot X) = \lambda \cdot \mathcal{D}(X)$ for any positive constant λ .
5. *Subadditivity*: If $X_1 \leq X_2$ almost surely, then $\mathcal{D}(X_1 + X_2) \leq \mathcal{D}(X_1) + \mathcal{D}(X_2)$.

We say that a measure is a *monetary distribution measure* if it satisfies the first three axioms. Furthermore, we call a distribution measure *coherent* if it satisfies all five axioms.

In practice, risk-management systems typically compute several measures for the same portfolio in order to capture different aspects of tail behavior. Having a unified definition for both risk and gain measures allows a portfolio manager to use the same tools to analyze both sides of a portfolio distribution in a coherent way.

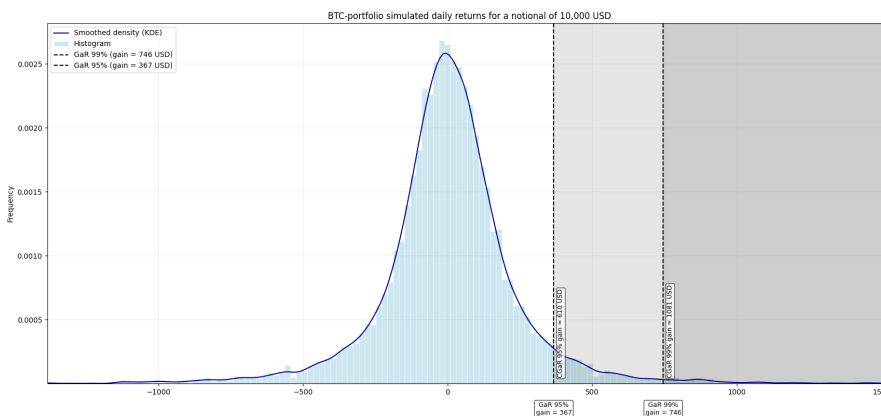


Figure 3: Gain at Risk (GaR) and Conditional Gain at Risk (CGaR) for the portfolio fully invested in Bitcoin (BTC).

3.3 Common distribution measures

Value at Risk The most widely used risk measure in practice is the *Value at Risk* (VaR). Given a confidence level ℓ with $0 < \ell < 1$, Value at Risk is defined as the smallest loss such that the probability of observing a loss greater than or equal to this value is at most $1 - \ell$. Equivalently, Value at Risk is the negative of the $(1 - \ell)$ empirical lower quantile of the sample returns.

Given a random variable X representing the portfolio return, we write $\text{VaR}^\ell(X)$ for the Value at Risk at level ℓ . In Figure 1 we show the distribution of the portfolio return for a portfolio fully invested in Bitcoin (BTC). More details on the simulations are provided later. Intuitively, the Value at Risk at level $\ell = 95\%$ can be visualized as the leftmost vertical line that leaves 5% of the return mass on its left side and 95% on its right side.

Conditional Value at Risk Another commonly used risk measure is the *Conditional Value at Risk* (CVaR), also called *Expected Shortfall*. It is defined as the expected loss in the scenarios that are worse than or equal to the Value at Risk. For a given level ℓ with $0 < \ell < 1$, we can write

$$\text{CVaR}^\ell(X) = -\mathbb{E}[X \mid X \leq -\text{VaR}^\ell(X)],$$

so that $\text{CVaR}^\ell(X)$ is reported as a positive cash amount when the position is risky. In Figure 2 we show the distribution of the portfolio return for a portfolio fully invested in Ether (ETH). Graphically, the Conditional Value at Risk at level $\ell = 95\%$ can be represented as the average of the losses (negative returns) that are below the VaR threshold at the same level (in the figure, the shaded areas).

Gain at Risk and Conditional Gain at Risk We now apply the gain-construction discussed above to the specific case where the underlying risk measures are VaR and CVaR. This leads to the *Gain at Risk* (GaR) and the *Conditional Gain at Risk* (CGaR).

Let ℓ be a confidence level with $0 < \ell < 1$. For any portfolio $X \in \mathcal{L}$ we define the Gain at Risk at level ℓ by

$$\text{GaR}^\ell(X) = \text{VaR}^\ell(-X).$$

Intuitively, GaR^ℓ is an upper-quantile threshold: it is the *rightmost* vertical line such that at least a fraction ℓ of the probability mass lies to its left (equivalently, at most $1 - \ell$ lies to its right). Similarly, we define the Conditional Gain at Risk by

$$\text{CGaR}^\ell(X) = \text{CVaR}^\ell(-X).$$

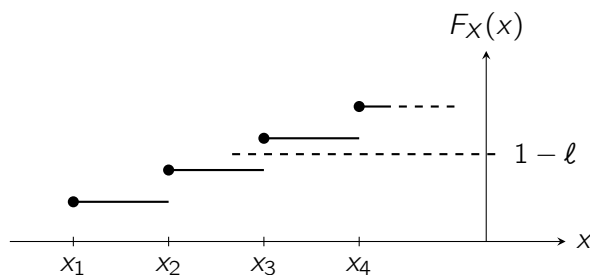


Figure 4: Discrete sample VaR. In this case $k=3$, so that the Value at Risk is $-x_3$.

Thus, GaR and CGaR measure the upside potential of X by applying the usual VaR and CVaR formulas to the short position $-X$.

Figures 3 and 5 illustrate examples of GaR and CGaR for portfolios fully invested in Bitcoin (BTC) and Ethereum (ETH), respectively. In the figures, GaR is represented as the rightmost vertical line that leaves $1 - \ell$ of the return on the its left side and ℓ on the right side. Similarly, CGaR is represented as the average of the gains (positive returns) that are above the GaR threshold at the same level (in the figure, the shaded areas).

4 Finite-Sample Distortion Measures

4.1 Numerical simulations and the finite-sample formulation

While the theory of risk measures is developed for general random variables, in practice we often work with a finite sample of portfolio returns. This is the case, for example, when portfolio returns are generated as scenarios of a numerical simulation.

Consider therefore a sample of portfolio returns x_1, \dots, x_n . Without loss of generality, we assume that the sample is sorted in ascending order, that is, $x_1 \leq \dots \leq x_n$. We denote by \mathcal{S}^n the space of all ordered samples of portfolio returns (the S in \mathcal{S}^n stands for sorted). In other words,

$$\mathcal{S}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \leq \dots \leq x_n\}.$$

With this ordering, it is easier to write explicit formulas for the distribution measures considered below. In this finite-sample setting, a risk measure ρ is simply a function $\rho : \mathcal{S}^n \rightarrow \mathbb{R}$.

4.2 Value at Risk and Conditional VaR

Value at Risk In the finite-sample setting, Value at Risk (VaR) is defined as the negative of the $(1 - \ell)$ -empirical lower quantile of the sample. Let $\mathbf{X} = (x_1, \dots, x_n) \in \mathcal{S}^n$ be the ordered sample of returns. Then the VaR at level ℓ is given by

$$\text{VaR}^\ell(\mathbf{X}) = -x_k, \quad (2)$$

where k is the integer such that

$$k - 1 < n(1 - \ell) \leq k.$$

Explicitly, k can be computed as

$$k = \lceil n(1 - \ell) \rceil, \quad (3)$$

where $\lceil z \rceil$ denotes the ceiling function, that is, the smallest integer greater than or equal to z .

Figure 4 illustrates the computation of the VaR for a small sample of returns.

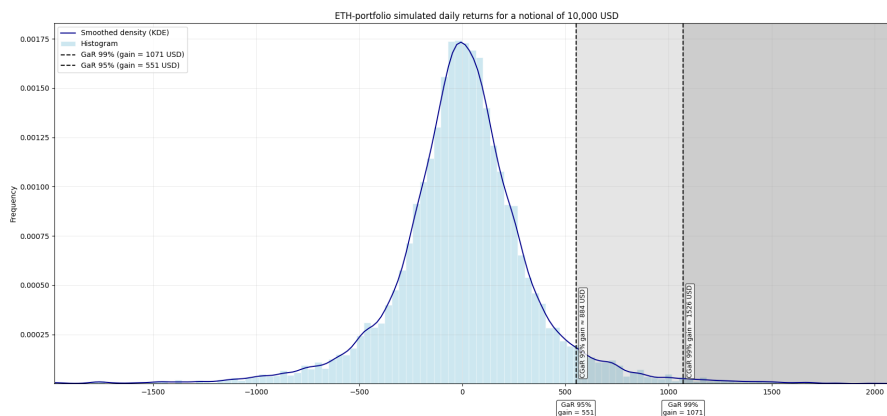


Figure 5: Gain at Risk (GaR) and Conditional Gain at Risk (CGaR) for the portfolio fully invested in Ethereum (ETH).

As an example, take $\ell = 0.99$ and $n = 999$. Then

$$n(1 - \ell) = 999 \cdot 0.01 = 9.99,$$

so that $k = 10$. In this case, the Value at Risk is the 10th worst return in the sample.

If we keep $\ell = 0.99$ but increase the sample size to $n = 1001$, then

$$k = \lceil n(1 - \ell) \rceil = \lceil 1001 \cdot 0.01 \rceil = \lceil 10.01 \rceil = 11,$$

so the Value at Risk becomes the 11th worst return in the sample. This simple example shows that VaR can change quickly when a few scenarios are added or removed, even if they are *not* in the tail of the distribution.

It can be shown that Value at Risk satisfies the axioms of a monetary risk measure introduced in the previous section, and it is also positively homogeneous. However, VaR in general does *not* satisfy the subadditivity axiom, hence VaR is *not* a coherent risk measure. This also means that VaR, in general, is not a convex risk measure (i.e. it does not play well with optimizations).

Conditional Value at Risk The *Conditional Value at Risk* (CVaR), also known as *Expected Shortfall*, is defined as the expected loss in the scenarios that are worse than or equal to the Value at Risk. For a given level ℓ with $0 < \ell < 1$ and ordered sample $\mathbf{X} = (x_1, \dots, x_n) \in \mathcal{S}^n$, the Conditional Value at Risk can be computed as

$$\text{CVaR}^\ell(\mathbf{X}) = \frac{-1}{1 - \ell} \left[\Delta \cdot x_k + \frac{1}{n} \sum_{i=1}^k x_i \right], \quad (4)$$

where k is the integer in equation (3) and Δ is defined as

$$\Delta = (1 - \ell) - \frac{k}{n}. \quad (5)$$

The quantity Δ measures the difference between the tail probability $1 - \ell$ and the proportion k/n of scenarios included in the tail. Since $k/n \geq 1 - \ell$ by construction, Δ is typically non-positive.

As an example, take $\ell = 0.99$, as earlier, and $n = 1000$. Then $n(1 - \ell) = 10$, so that $k = 10$ and $\Delta = 0$. In this case, the CVaR is simply the average of the 10 worst returns in the sample, that is, the average of the scenarios whose losses are greater than or equal to the VaR.

It can be shown that Conditional Value at Risk satisfies all the axioms of both monetary and coherent risk measures, including subadditivity. Hence CVaR is also a convex risk measure, i.e. it plays well with optimizations.

4.3 Gain at Risk and Conditional GaR

Gain at Risk In the finite-sample setting, *Gain at Risk* (GaR) is defined as the Value at Risk of the short position, i.e. $\text{GaR}^\ell(\mathbf{X}) = \text{VaR}^\ell(-\mathbf{X})$. It can be shown that (see appendix A.4) equivalently GaR is the empirical *upper* quantile of the return sample: it reports a gain threshold such that only a fraction $1 - \ell$ of scenarios exceed it.

Let $\mathbf{X} = (x_1, \dots, x_n) \in \mathcal{S}^n$ be the ordered sample of returns, with $x_1 \leq \dots \leq x_n$. Define the index k as in equation (3) and j as

$$j = n - k + 1 \quad (6)$$

Then the Gain at Risk at level ℓ can be computed as

$$\text{GaR}^\ell(\mathbf{X}) = x_j.$$

Thus, $\text{GaR}^\ell(\mathbf{X}) = x_j$ is the j -th order statistic (equivalently, the k -th largest return), where $k = \lceil n(1 - \ell) \rceil$.

As an example, take $\ell = 0.99$ and $n = 999$ as above, so that $k = \lceil 999 \cdot 0.01 \rceil = 10$ and $j = 999 - 10 + 1 = 990$. Then $\text{GaR}^{0.99}(\mathbf{X}) = x_{990}$: the gain threshold is the 10th best return in the sample.

If we keep $\ell = 0.99$ but increase the sample size to $n = 1001$, then $k = \lceil 1001 \cdot 0.01 \rceil = 11$ and $j = 1001 - 11 + 1 = 991$, so the GaR becomes $x_{991} = x_j$. As for VaR, this illustrates that GaR can change when scenarios are added or removed, reflecting the discreteness of empirical quantiles.

Since $\text{GaR}^\ell(\mathbf{X}) = \text{VaR}^\ell(-\mathbf{X})$ and VaR is a monetary risk measure, it follows that GaR is a monetary *gain* measure (in particular, it is cash-additive with a + sign, and is positively homogeneous). However, because VaR is not subadditive in general, GaR also does not satisfy subadditivity in general, and therefore is not coherent as a distribution measure (and is not convex in general).

Conditional Gain at Risk The *Conditional Gain at Risk* (CGaR) is defined analogously as the Conditional Value at Risk of the short position, i.e. $\text{CGaR}^\ell(\mathbf{X}) = \text{CVaR}^\ell(-\mathbf{X})$. It can be interpreted as the expected gain in the scenarios that are better than or equal to the Gain at Risk threshold.

Let $\ell \in (0, 1)$ and $\mathbf{X} = (x_1, \dots, x_n) \in \mathcal{S}^n$ be the ordered sample. Let k be defined as in equation (3), Δ as in equation (5) and j as in equation (6). Then CGaR at level ℓ can be computed in the finite-sample setting as

$$\text{CGaR}^\ell(\mathbf{X}) = \frac{1}{1 - \ell} \left[\Delta \cdot x_j + \frac{1}{n} \sum_{i=j}^n x_i \right].$$

Here the summation runs over the k best scenarios and the term $\Delta \cdot x_j$ plays the same role as the term $\Delta \cdot x_k$ in the CVaR formula: it adjusts for the fact that k/n may exceed the tail probability $1 - \ell$ in the discrete sample.

As an example, take $\ell = 0.99$ and $n = 1000$, so that $k = 10$, $j = 991$ and $\Delta = 0$. Then

$$\text{CGaR}^{0.99}(\mathbf{X}) = \frac{1}{10} \sum_{i=991}^{1000} x_i,$$

i.e. CGaR is simply the average of the 10 best returns in the sample.

Since $\text{CGaR}^\ell(\mathbf{X}) = \text{CVaR}^\ell(-\mathbf{X})$ and CVaR is coherent, it follows that CGaR satisfies the gain-measure counterparts of the coherence axioms (in particular, cash-additivity with a + sign, positive homogeneity, and subadditivity). Hence CGaR is a coherent (and therefore convex) distribution *gain* measure, and it behaves well in optimization-based applications.

4.4 Distortion measures

In the previous sections we consider the generic definition of distribution measures. Several risk measures can be created using those generic definitions. However, in the specific case of numerical simulations there is a subset of distribution measures that are particularly useful. These distribution measures are called *distortion measures* and they are characterized by the fact that they are computed as a linear combination of the portfolio returns.

In appendix A we provide a formal definition of distortion measures. However, that definition is highly technical and can be greatly simplified in the specific case of a finite-sample of portfolio returns.

Distortion measures in the finite-sample setting

In the finite-sample framework, portfolio returns are represented by an ordered sample $\mathbf{X} \in \mathcal{S}^n$. A *distortion measure* on this finite-sample space is any functional $\mathcal{D}(\mathbf{X})$ that can be written as a linear form in the ordered sample,

$$\mathcal{D}(\mathbf{X}) = \sum_{i=1}^n W_i x_i, \quad (7)$$

for some vector

$$\mathbf{W} = (W_1, \dots, W_n)^T \in \mathbb{R}^n,$$

that we call the *weight vector* of \mathcal{D} .

For monetary, cash-additive risk measures, the weights satisfy

$$\sum_{i=1}^n W_i = -1, \quad (8)$$

so that adding a sure amount of cash c to all scenarios shifts the value of $\mathcal{D}(\mathbf{X})$ by $-c$. Conversely, for gain measures the weights satisfy

$$\sum_{i=1}^n W_i = 1, \quad (9)$$

so that adding a sure amount of cash c to all scenarios shifts the value of $\mathcal{D}(\mathbf{X})$ by $+c$. In general we have that

$$\sum_{i=1}^n W_i = m_s,$$

where m_s is the measure sign.

Specific examples of distortion risk measures

Looking at the finite-sample definition (2) of VaR and the definition (4) of CVaR, we see that both can be written as linear forms of the ordered sample \mathbf{X} , with weights that depend only on the confidence level ℓ and on the sample size n .

In other words, we can write VaR as

$$\text{VaR}^\ell(\mathbf{X}) = \mathbf{W}^{\text{VaR}} \odot \mathbf{X},$$

where the weight vector \mathbf{W}^{VaR} is defined componentwise by

$$\mathbf{W}_i^{\text{VaR}} = \begin{cases} 0, & i < k, \\ -1, & i = k, \\ 0, & i > k, \end{cases} \quad (10)$$

and k is the integer defined in equation (3). In the above equation, the symbol \odot denotes the scalar (dot) product of the weight vector \mathbf{W}^{VaR} and the ordered sample \mathbf{X} . More explicitly, for any two vectors $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^n$ we set

$$\mathbf{X} \odot \mathbf{Y} = \sum_{i=1}^n X_i Y_i.$$

Note that the weight vector \mathbf{W}^{VaR} sums to -1 , which is consistent with the cash-additivity property of VaR.

For Conditional Value at Risk we obtain a similar representation:

$$\text{CVaR}^\ell(\mathbf{X}) = \mathbf{W}^{\text{CVaR}} \odot \mathbf{X},$$

where the weight vector \mathbf{W}^{CVaR} is defined by

$$\mathbf{w}_i^{\text{CVaR}} = \begin{cases} -\frac{1}{n(1-\ell)}, & i < k, \\ -\frac{1}{n(1-\ell)} - \frac{\Delta}{1-\ell}, & i = k, \\ 0, & i > k, \end{cases} \quad (11)$$

and Δ is defined as in equation (5). Again the weight vector \mathbf{W}^{CVaR} sums to -1 , in line with the cash-additivity property of CVaR.

Writing a risk measure as a linear form in the ordered sample \mathbf{X} is particularly useful when we work with a large number of scenarios, because it allows us to compute the risk measure in a fully vectorized way. This motivates the study of the class of risk measures that admit such a representation.

Gain at Risk and Conditional Gain at Risk. Since $\text{GaR}^\ell(\mathbf{X}) = \text{VaR}^\ell(-\mathbf{X})$ and $\text{CGaR}^\ell(\mathbf{X}) = \text{CVaR}^\ell(-\mathbf{X})$, both GaR and CGaR are distortion *gain* measures and therefore admit weight-vector representations analogous to equations (10) and (11).

Again, let j be defined as in equation (6), then

$$\text{GaR}^\ell(\mathbf{X}) = \mathbf{W}^{\text{GaR}} \odot \mathbf{X}, \quad \text{CGaR}^\ell(\mathbf{X}) = \mathbf{W}^{\text{CGaR}} \odot \mathbf{X},$$

with weight vectors given component-wise by

$$\mathbf{w}_i^{\text{GaR}} = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases}$$

and

$$\mathbf{w}_i^{\text{CGaR}} = \begin{cases} 0, & i < j, \\ \frac{1}{n(1-\ell)} + \frac{\Delta}{1-\ell}, & i = j, \\ \frac{1}{n(1-\ell)}, & i > j, \end{cases}$$

where Δ was defined earlier in equation (5).

In particular, we have

$$\sum_{i=1}^n \mathbf{w}_i^{\text{GaR}} = +1 \quad \text{and} \quad \sum_{i=1}^n \mathbf{w}_i^{\text{CGaR}} = +1,$$

in agreement with the cash-additivity normalization (9) for gain measures.

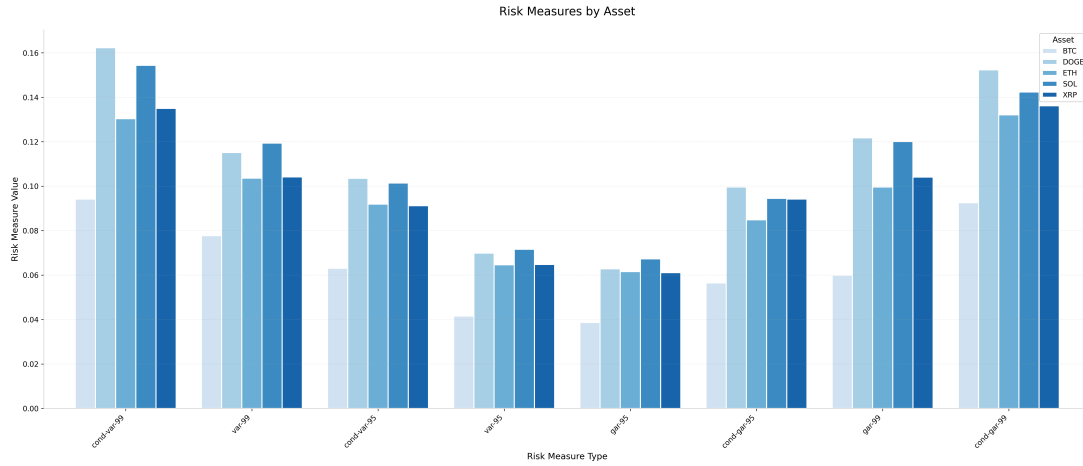


Figure 6: VaR, CVaR, GaR, and CGaR at confidence levels 95% and 99% for a set of sample tokens.

4.5 Equivalent weight representations of distortion measures

In the finite-sample setting, a distortion measure is thus completely described by its weight vector \mathbf{W} through (7). However, this representation may not give rise to properties such as smoothness with respect to the confidence level. Hence, we introduce the concept of equivalent weight vectors.

Let $\tilde{X} \in \mathcal{S}^n$ be a specific ordered sample (for example, the samples generated by a particular simulation procedure). We say that a weight vector $\tilde{\mathbf{W}}$ is *equivalent* to \mathbf{W} for the distortion measure \mathcal{D} with respect to the sample \tilde{X} if

$$\mathcal{D}(\tilde{X}) = \tilde{\mathbf{W}} \odot \tilde{X} = \mathbf{W} \odot \tilde{X} \quad (12)$$

Equivalent weight vectors can be used to modify or enhance the qualitative properties of a given risk measure (for example, smoothness of the weights or additional constraints on their magnitudes) without changing its numerical value on the relevant sample.

Given an equivalent weight vector $\tilde{\mathbf{W}}$ we note that the functional

$$\mathbf{X} \rightarrow \tilde{\mathbf{W}} \odot \mathbf{X}, \quad (13)$$

is a linear operator in the space \mathbb{R}^n (not just in the space \mathcal{S}^n). This observation will be useful in future work—for example, when computing marginal contributions for discrete distortion measures; see reference [11].

4.6 Vectorized computation of multiple distortion measures

In many applications we want to compute several distortion measures $\mathcal{D}_1, \dots, \mathcal{D}_m$ at the same time. Assume that each distortion measure \mathcal{D}_j is defined by a weight vector $\mathbf{W}_j \in \mathbb{R}^n$ through

$$\mathcal{D}_j(\mathbf{X}) = \mathbf{W}_j \odot \mathbf{X},$$

where \odot denotes the scalar (dot) product in \mathbb{R}^n . We collect the weight vectors as columns of the matrix

$$\mathcal{W} = [\mathbf{W}_1, \dots, \mathbf{W}_m] \in \mathbb{R}^{n \times m}.$$

We define the vector of distortion-measure values by

$$\mathbf{D}(\mathbf{X}) = (\mathcal{D}_1(\mathbf{X}), \dots, \mathcal{D}_m(\mathbf{X}))^\top \in \mathbb{R}^m.$$

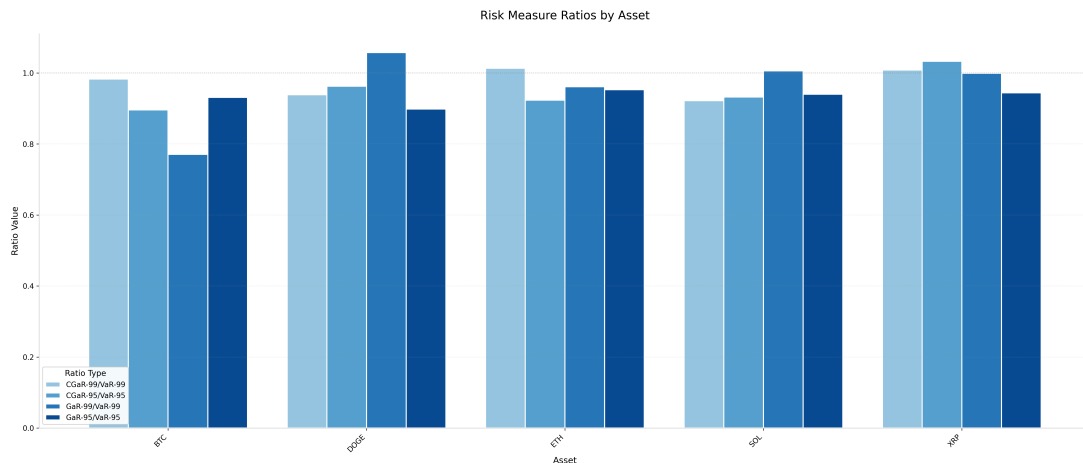


Figure 7: Ratios CGaR/CVaR and GaR/VaR at confidence levels 95% and 99% for the same set of tokens as in figure 6.

Using the linear representation, we can write

$$\mathbf{D}(\mathbf{X}) = \mathcal{W}^T \mathbf{X},$$

which is a standard matrix–vector multiplication. The j -th entry of $\mathbf{D}(\mathbf{X})$ is then

$$\mathcal{D}_j(\mathbf{X}) = \mathbf{W}_j \odot \mathbf{X},$$

as expected.

This matrix formulation is very convenient in vectorized implementations: the matrix \mathcal{W} can be precomputed once (for given confidence levels, sample size, and types of distortion measures) and then reused many times for different samples \mathbf{X} , for example in large-scale simulation studies or when recomputing distortion measures across many portfolios.

4.7 Numerical illustration

To illustrate the practical use of the above concepts, we now present a numerical example. We compute VaR, CVaR, GaR, and CGaR for the tokens Bitcoin (BTC), Dogecoin (DOGE), Ethereum (ETH), Solana (SOL), and Ripple (XRP), at confidence levels 95% and 99%, using a sample of approximately 12,000 simulated one-day cash-return scenarios for each token. The details of the simulation procedure are provided in reference [12].

Figure 6 reports the values obtained for VaR, CVaR, GaR, and CGaR for all tokens and confidence levels. As expected, the values of both downside risk measures (VaR, CVaR) and upside gain measures (GaR, CGaR) increase with the confidence level. Moreover, the conditional measures (CVaR and CGaR) are always larger than their non-conditional counterparts (VaR and GaR), reflecting the fact that they average over the worst (for losses) or best (for gains) scenarios beyond the corresponding quantile.

To compare risk and gain measures more directly, we consider the ratios of gain measures to the corresponding risk measures. For a given token and confidence level ℓ , we define

$$\text{GaR/VaR-ratio}_\ell = \frac{\text{GaR}^\ell(X)}{\text{VaR}^\ell(X)}, \quad \text{CGaR/CVaR-ratio}_\ell = \frac{\text{CGaR}^\ell(X)}{\text{CVaR}^\ell(X)}.$$

Figure 7 shows these ratios at confidence levels 95% and 99% for all tokens considered above. We observe that these ratios are almost always less than 1, indicating that, for most tokens and levels,

the potential gain (as measured by GaR and CGaR) is smaller than the corresponding potential loss (as measured by VaR and CVaR). This illustrates how distribution measures of risk and gain type can be used jointly to assess the asymmetry between downside risk and upside potential.

5 Conclusion

In this paper we developed an intrinsically finite-sample perspective on monetary and coherent functionals of portfolio returns. Working directly with ordered scenario outcomes, we showed that VaR, CVaR, and a broad class of distortion-type measures admit linear representations in the order statistics and are therefore fully characterized by scenario-weight vectors. In this representation, core axioms translate into transparent constraints on the weights—most notably, cash-additivity corresponds to the normalization $\sum_i W_i = -1$ for risk measures.

We then constructed upside *gain measures* by applying risk measures to the short position. This yields gain functionals whose finite-sample weights are obtained by a simple transformation and sum to +1, making the symmetry between downside and upside explicit. This motivates the unified notion of *distribution measures*, within which VaR/CVaR and their gain counterparts GaR/CGaR can be handled with the same finite-sample and computational machinery. From a numerical standpoint, the weight-vector viewpoint leads to implementation-ready matrix formulas that evaluate many measures at once over large scenario panels. The digital-asset illustration shows how these measures behave at common confidence levels and how ratios such as GaR/VaR and CGaR/CVaR provide a compact way to quantify tail asymmetry between gains and losses.

Several extensions follow naturally. On the methodological side, the freedom provided by equivalent weight vectors can be exploited to design smoother or more robust versions of existing measures without changing their reported values on the scenario sets used for measurement. On the applications side, the same finite-sample calculus can be embedded in portfolio construction, backtesting and stress testing, and dynamic monitoring pipelines for digital assets and mixed portfolios. Overall, we view the weight-based finite-sample formulation as a practical complement to the abstract theory of risk measures in modern simulation-driven risk management.

The practical takeaway is that (many) risk and gain measures can be implemented as fixed weight vectors applied to sorted scenarios, making large-scale evaluation a single matrix multiply.

A General Distributional Definition of Distortion Measures

For completeness, this appendix records a distributional (probabilistic) definition of distortion measures and makes explicit how the gain measure associated with a distortion risk measure can be expressed in terms of upper quantiles. This material is more technical than the main text and can be skipped on a first reading. For additional background on law-invariant (distribution-based) risk measures and related representations, see, e.g., references [5, 7, 9, 10].

A.1 Probabilistic definition of distortion risk measures

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let \mathcal{L} be a linear space of integrable random variables X representing future portfolio returns. For $X \in \mathcal{L}$, we denote by F_X its cumulative distribution function (cdf),

$$F_X(x) = \mathbb{P}(X \leq x), \quad x \in \mathbb{R}.$$

The *lower quantile function*, a generalized inverse of the cdf, is defined as

$$Q_X(u) = \inf\{x \in \mathbb{R} : F_X(x) \geq u\}, \quad u \in (0, 1), \quad (14)$$

and the *upper quantile function* is defined as

$$\widehat{Q}_X(u) = \sup\{x \in \mathbb{R} : F_X(x) \leq u\}, \quad u \in (0, 1). \quad (15)$$

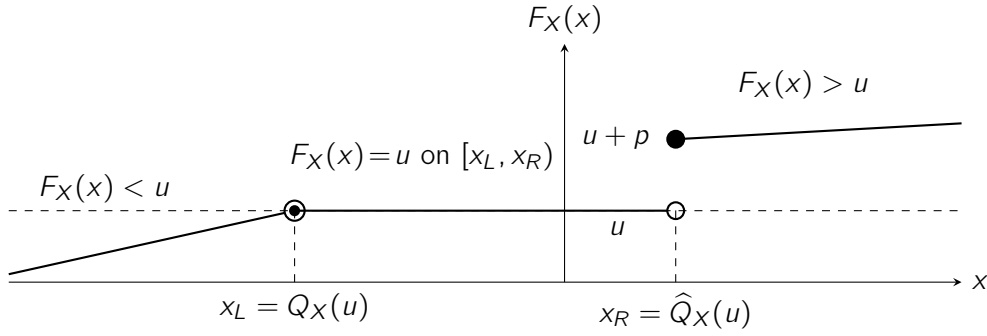


Figure 8: Lower and upper quantiles at a probability level u where the cdf has a plateau. In this example, $F_X(x) = u$ for $x \in [x_L, x_R)$, so the inverse image of u is an interval. The lower quantile selects the left endpoint, $Q_X(u) = x_L$, while the upper quantile selects the right endpoint, $\widehat{Q}_X(u) = x_R$ (as a supremum). This situation is typical for discrete or mixed distributions.

If F_X is continuous and strictly increasing at level u , then $Q_X(u) = \widehat{Q}_X(u)$. When F_X is not strictly increasing (as is typical for discrete or mixed distributions), the inverse at some probability levels can be set-valued; in that case, $Q_X(u)$ and $\widehat{Q}_X(u)$ select the left and right endpoint (conventionally) of the inverse image.

Figure 8 illustrates this distinction in the case where the cdf has a plateau at level u : here $F_X(x) = u$ for $x \in [x_L, x_R)$, so $Q_X(u) = x_L$ while $\widehat{Q}_X(u) = x_R$ (the latter as a supremum, which may be attained only as a left limit when there is a jump at x_R). Note that in the case of figure 8, we can compute the following probability:

$$\mathbb{P}(X \in [x_L, x_R)) = F_X(x_R) - F_X(x_L) = u + p - u = p,$$

i.e. the probability of the event $X \in [x_L, x_R)$ is equal to p . In the discrete case with uniform distribution, we have $p = 1/n$ for a sample of size n .

A *distortion risk measure* is a monetary, law-invariant risk measure $\rho : \mathcal{L} \rightarrow \mathbb{R}$ that can be represented as a signed quantile-weighted functional: there exists a finite signed measure μ_ρ on $(0, 1]$ such that

$$\rho(X) = \int_0^1 Q_X(u) d\mu_\rho(u), \quad X \in \mathcal{L}. \quad (16)$$

The measure μ_ρ plays the role of a *weight measure* over probability levels and is the continuous analogue of the finite-sample weight vectors used in the main text.

Cash-additivity forces a normalization of the total mass of μ_ρ . Indeed, since $Q_{X+c}(u) = Q_X(u) + c$ for any real-valued constants c , we have

$$\rho(X + c) = \int_0^1 [Q_X(u) + c] d\mu_\rho(u) = \rho(X) + c \mu_\rho((0, 1]).$$

Therefore, to match the usual sign convention $\rho(X + c) = \rho(X) - c$, the weight measure must satisfy

$$\mu_\rho((0, 1]) = -1. \quad (17)$$

Weight functions. When μ_ρ is absolutely continuous with respect to Lebesgue measure, it admits a density W_ρ such that

$$d\mu_\rho(u) = W_\rho(u) du, \quad (18)$$

and (16) becomes

$$\rho(X) = \int_0^1 Q_X(u) W_\rho(u) du.$$

More generally, allowing W_ρ to be a generalized function (for example, a Dirac delta) covers the case where μ_ρ has singular components. In this representation, (17) becomes

$$\int_0^1 W_\rho(u) du = -1,$$

mirroring the finite-sample constraint $\sum_i W_i = -1$ for discrete distortion *risk* measures.

A.2 Value at Risk and CVaR in quantile-weight form

Value at Risk. For Value at Risk at level $\ell \in (0, 1)$,

$$\text{VaR}^\ell(X) = -Q_X(1 - \ell) = \int_0^1 Q_X(u) W_{\text{VaR}}^\ell(u) du,$$

with generalized weight function W_{VaR}^ℓ defined as

$$W_{\text{VaR}}^\ell(u) = -\delta_{1-\ell}(u), \quad (19)$$

Note that we have

$$\int_0^1 W_{\text{VaR}}^\ell(u) du = -\int_0^1 \delta_{1-\ell}(u) du = -1.$$

Conditional Value at Risk (Expected Shortfall). For Conditional Value at Risk at level $\ell \in (0, 1)$, we have,

$$\text{CVaR}^\ell(X) = -\frac{1}{1-\ell} \int_0^{1-\ell} Q_X(u) du = \int_0^1 Q_X(u) W_{\text{CVaR}}^\ell(u) du,$$

with weight function W_{CVaR}^ℓ defined as

$$W_{\text{CVaR}}^\ell(u) = -\frac{1}{1-\ell} \mathbf{1}_{(0, 1-\ell]}(u). \quad (20)$$

Again, we have

$$\int_0^1 W_{\text{CVaR}}^\ell(u) du = -\int_0^1 \left[\frac{1}{1-\ell} \mathbf{1}_{(0, 1-\ell]}(u) \right] du = -\frac{1}{1-\ell} \int_0^{1-\ell} 1 du = -1.$$

A.3 Gain measures induced by distortion risk measures

In Section 3 we defined the gain measure associated with a risk measure ρ by applying ρ to the short position:

$$\gamma_\rho(X) = \rho(-X).$$

If ρ admits the quantile-weight representation (16), then

$$\gamma_\rho(X) = \int_0^1 Q_{-X}(u) d\mu_\rho(u) = \int_0^1 Q_{-X}(u) W_\rho(u) du.$$

Consider now the following *flipped-quantile identity*, proved in the next subsection:

$$Q_{-X}(u) = -\widehat{Q}_X(1 - u), \quad u \in (0, 1), \quad (21)$$

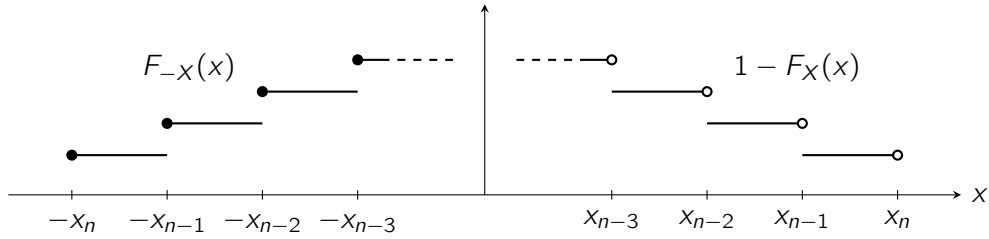


Figure 9: Illustration of the flipped-quantile identity. On the left, the cdf $F_{-X}(x)$ of the random variable $-X$ is shown, with the lower quantile $Q_Y(u)$ denoted by a filled dot. On the right, the survival function $1 - F_X(x)$ is shown, with the upper quantile $\widehat{Q}_X(u)$ at a given probability level u denoted by an open dot. In this figure the x_i 's are equally spaced, however in the general case they do not need to be.

where Q_{-X} is the lower quantile function of $-X$ defined in (14) and \widehat{Q}_X is the upper quantile function of X defined in (15). In the specific case of continuous distributions, we have $Q_X = \widehat{Q}_X$ and equation (21) reduces to $Q_{-X}(u) = -Q_X(1 - u)$.

We apply now the flipped-quantile identity to the gain measure $\gamma_\rho(X)$:

$$\gamma_\rho(X) = - \int_0^1 \widehat{Q}_X(1 - u) W_\rho(u) du.$$

With the substitution $t = 1 - u$ (so $u = 1 - t$ and $du = -dt$), this becomes

$$\gamma_\rho(X) = - \int_0^1 \widehat{Q}_X(t) W_\rho(1 - t) dt = \int_0^1 \widehat{Q}_X(t) \widetilde{W}_\rho(t) dt,$$

where we define the *upside weight density* (in generalized sense) by

$$\widetilde{W}_\rho(t) = -W_\rho(1 - t), \quad t \in (0, 1]. \quad (22)$$

Thus the gain measure induced by a distortion risk measure is itself a distortion-type functional, but acting on upper quantiles with a flipped and sign-changed weight. Moreover,

$$\int_0^1 \widetilde{W}_\rho(t) dt = - \int_0^1 W_\rho(1 - t) dt = - \int_0^1 W_\rho(u) du = 1,$$

which matches the finite-sample normalization $\sum_i W_i = 1$ for gain measures.

A.4 Proof of the flipped-quantile identity

In this subsection we prove the flipped-quantile identity, i.e. equation (21), i.e. ,

$$Q_{-X}(u) = -\widehat{Q}_X(1 - u), \quad u \in (0, 1).$$

Figure 9 illustrates the flipped-quantile identity (look at the figure caption for the details). Let X be a real-valued random variable with cumulative distribution function (cdf)

$$F_X(x) = \mathbb{P}(X \leq x), \quad x \in \mathbb{R}.$$

We assume the standard properties: F_X is non-decreasing, right-continuous with left limits (càdlàg), and satisfies $\lim_{x \rightarrow -\infty} F_X(x) = 0$ and $\lim_{x \rightarrow \infty} F_X(x) = 1$.

Recall the lower and upper quantile functions (generalized inverses) of X :

$$\begin{aligned} Q_X(u) &= \inf\{x \in \mathbb{R} : F_X(x) \geq u\}, & u \in (0, 1), \\ \widehat{Q}_X(u) &= \sup\{x \in \mathbb{R} : F_X(x) \leq u\}, & u \in (0, 1). \end{aligned}$$

Let $Y := -X$. Its cdf satisfies, for any $y \in \mathbb{R}$,

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(-X \leq y) = \mathbb{P}(X \geq -y) = 1 - \mathbb{P}(X < -y) = 1 - F_X(-y^-), \quad (23)$$

where $F_X(a^-) := \lim_{x \uparrow a} F_X(x)$ denotes the left limit of F_X at a .

Proposition 1 (Flipped-quantile identity) For every $u \in (0, 1)$,

$$Q_{-X}(u) = -\widehat{Q}_X(1 - u).$$

Proof. Fix $u \in (0, 1)$ and define

$$Z := \widehat{Q}_X(1 - u) = \sup\{x \in \mathbb{R} : F_X(x) \leq 1 - u\}.$$

We show that $-Z$ is the lower u -quantile of $Y = -X$, i.e. that it is the smallest¹ y such that $F_Y(y) \geq u$.

Step 1: show that $F_Y(-Z) \geq u$. By definition of Z as a supremum, we have

$$F_X(Z^-) \leq 1 - u \iff -[1 - F_X(Z^-)] \leq -u \iff 1 - F_X(Z^-) \geq u,$$

otherwise, if $F_X(Z^-) > 1 - u$, then some values immediately to the left of Z would satisfy $F_X(x) > 1 - u$, contradicting the definition of Z as a supremum. Hence, using (23) with $y = -Z$, we have

$$F_Y(-Z) = 1 - F_X(Z^-) \geq u.$$

Step 2: If $y < -Z$, then $F_Y(y) < u$. Let $y < -Z$ and define $z = -y$, so $z > Z$. Since $Z = \sup\{x : F_X(x) \leq 1 - u\}$, it follows that for every $x > Z$ we must have $F_X(x) > 1 - u$. In particular, because F_X is non-decreasing,

$$F_X(z^-) = \sup_{x < z} F_X(x) \geq F_X(x_0) > 1 - u$$

for any x_0 with $Z < x_0 < z$. Therefore,

$$F_Y(y) = 1 - F_X(z^-) < u.$$

Conclusion. We have shown that $F_Y(-Z) \geq u$ and that $F_Y(y) < u$ for all $y < -Z$. By the definition of the lower quantile,

$$Q_Y(u) = \inf\{y \in \mathbb{R} : F_Y(y) \geq u\} = -Z.$$

Since $Y = -X$, this yields $Q_{-X}(u) = -\widehat{Q}_X(1 - u)$, which is (21). ■

¹Technically the infimum is required here, but because of the right-continuity of F_Y the infimum and the minimum coincide.

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