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ANALYSIS

The Enhanced Jump-Wing SVI Parameterization

April 2025

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https://www.talos.com/
Version 1.0

April 2025*

Abstract

Given the standard raw SVI parameterization, we look at an equivalent parameterization that is very similar to the well-known *Jump-Wing SVI parameterization*. We call this new parameterization the *Enhanced Jump-Wing SVI parameterization* and show that it can be inverted to recover the raw SVI parameters for a wider range of parameters compared to the traditional Jump-Wing SVI parameterization. We explicitly derive the formula to recover the raw SVI parameters from the Enhanced Jump-Wing SVI parameters and then provide a vectorized implementation of the formula.

1 Introduction

The SVI (Stochastic Volatility Inspired) parameterization is a popular way to represent the implied volatility surface. We call this parameterization the *raw* SVI model which is based on the standard five parameters: $\chi_{\text{raw}} = \{a, b, \rho, m, \sigma\}$. This parameterization is widely used in the literature and in practice. Given these five raw SVI parameters we can compute the total implied variance $w(k; \chi_{\text{raw}})$ for any strike k.

Following closely the paper of Gatheral and Jacquier, see reference [1], we introduce a Modified *Jump-Wing* SVI parameterization. This alternative parameterization has the following characteristics:

- It has an explicit dependence on the time to expiration;
- it has a more intuitive interpretation of the parameters;
- it would be constant if the smile perfectly scaled as $1/\sqrt{w_t}$;
- it is invertible for a wider range of parameters compared to the traditional Jump-Wing SVI parameterization.

We are not aware of any previous work that has introduced this parameterization.

In this paper we provide an alternative explicit derivation of the inverse formula to recover the raw SVI parameters from the enhanced Jump-Wing SVI parameters, that is a bit different from the one given in reference [1]. We also show how to compute the raw SVI parameters in a vectorized way, which is useful for real-time calibration of the SVI model.

*Latest review April 9, 2025

2 The raw SVI parameterization

For a given parameter set $\chi_{\text{raw}} = \{a, b, \rho, m, \sigma\}$, the *raw SVI parameterization* of total implied variance reads:

$$w(k; \chi_{\text{raw}}) = a + b\left\{\rho(k-m) + \sqrt{(k-m)^2 + \sigma^2}\right\}, \tag{1}$$

where $a \in \mathbb{R}$, $b \geq 0$, $|\rho| \leq 1$, $m \in \mathbb{R}$, $\sigma \geq 0$, and the condition $a + b\,\sigma\sqrt{1-\rho^2} \geq 0$, which ensures that $w(k; \chi_R) \geq 0$ for all $k \in \mathbb{R}$, i.e. the absence of static arbitrage. In other words, this condition ensures that the minimum of the function $w(\cdot; \chi_{\text{raw}})$ is non-negative.

Note further that the function $k \mapsto w(k; \chi_{\text{raw}})$ is strictly convex on the whole real line. Also, while traditionally the raw SVI parameters are defined for $|\rho| < 1$ and $\sigma > 0$, we allow for the case $\sigma = 0$ and $|\rho| = 1$, in order to have a more general parameterization.

In practice the five raw SVI parameters have the following effects:

- Increasing a increases the general level of variance, a vertical translation of the smile;
- Increasing b increases the slopes of both the put and call wings, tightening the smile;
- Increasing ρ decreases (increases) the slope of the left (right) wing, a counter-clockwise rotation of the smile;
- Increasing m translates the smile to the right;
- Increasing σ reduces the at-the-money (ATM) curvature of the smile.

As there are many references on the raw SVI parameterization, we will not go into further details here.

3 The Enhanced Jump-Wing SVI parameterization

Given the five raw SVI parameters, see equation (1), we define the enhanced SVI *Jump Wings* six parameters $\chi_{\text{ejw}} = \{v_t, \psi_t, p_t, c_t, \tilde{v}_t, \xi_t\}$ as follows:

$$w_t = t \cdot v_t = a + b\left(-\rho m + \sqrt{m^2 + \sigma^2}\right), \tag{2}$$

$$\psi_t = \frac{b}{2\sqrt{w_t}}\left(\rho - \frac{m}{\sqrt{m^2 + \sigma^2}}\right), \tag{3}$$

$$p_t = \frac{b\,(1-\rho)}{\sqrt{w_t}}, \tag{4}$$

$$c_t = \frac{b\,(1+\rho)}{\sqrt{w_t}}, \tag{5}$$

$$t\,\tilde{v}_t = \left(a + b\,\sigma\sqrt{1-\rho^2}\right) \tag{6}$$

$$\xi_t = \frac{b}{\sigma} \quad \text{if } \sigma > 0\,, \quad \xi_t = 0 \quad \text{if } \sigma = 0\,, \tag{7}$$

where t is the time to expiration according to some day-count convention. While the at-the-money total variance w_t is not an explicit parameter in the raw SVI parameterization, it is convenient to use it as a shorthand in the following computations. Note that when both m and ρ are zero, the volatility surface is symmetric so that we can set ψ_t to zero, regardless of the value of σ.

Since $b \geq 0$ and $|\rho| \leq 1$, we have that

$$p_t \geq 0 \quad \text{and} \quad c_t \geq 0$$

The enhanced SVI-JW parameters have the following interpretations:

- v_t gives the at-the-money variance;
- ψ_t gives the at-the-money skew;
- p_t gives the slope of the left (put) wing;
- c_t gives the slope of the right (call) wing;
- $\tilde{v}_t$ is the minimum implied variance.
- ξ_t is the shifted at-the-money total-variance convexity, i.e., the convexity at $k=m$.

Note that the ξ_t parameter is not present in the traditional Jump-Wing SVI parameterization and it is introduced here so that the Jump-Wing SVI parameterization is invertible over the whole codomain of the raw SVI parameters.

Interpolation and extrapolation of the SVI calibrations.

As mentioned earlier, in normal market conditions the χ_{ejw} parameters are expected to be almost constant. For this reason they are good candidates for the interpolation and extrapolation of the implied volatility surface.

For example, suppose we know the raw SVI parameters at times t_1 and t_2. Then we can compute the corresponding enhanced Jump-Wing SVI parameters at time t_1 and t_2 and interpolate them at any time t between t_1 and t_2. We then invert the interpolated Jump-Wing SVI parameters to recover the interpolated raw SVI parameters at time t.

Similarly, suppose we know the raw SVI parameters at time t_1 and want to extrapolate them to time $t < t_1$. We can compute the Jump-Wing SVI parameters at time t_1 and assume them to be also valid at time t. We then invert the Jump-Wing SVI parameters to recover the raw SVI parameters at time t. Similarly for the case when $t > t_2$.

Note that, because the raw SVI parameters are not invariant under time transformation, the direct interpolation and extrapolation of the raw SVI parameters is not advisable, as it is not equivalent to the interpolation and extrapolation of the Jump-Wing SVI parameters. All these considerations are applicable because the transformation from the raw SVI parameters to the Jump-Wing SVI parameters is highly non-linear.

In the following sections we assume to have a realization of the enhanced Jump-Wing SVI parameters $\chi_{\text{ejw}}=\{v_t, \psi_t, p_t, c_t, \tilde{v}_t, \xi_t\}$ and we want to recover the raw SVI parameters $\chi_{\text{raw}}=\{a, b, \rho, m, \sigma\}$ at the time to expiration t. Note that, as customary in the literature, we keep the subscript t in the enhanced JW parameters however we do not write it in the raw SVI parameters, as the time to expiration is always assumed to be known. Also we only consider values of the χ_{ejw} parameters that are in the codomain of the raw SVI parameters, i.e., χ_{ejw} is always obtained from a valid raw SVI parameter set.

4 Computation of the b and ρ parameters

In order to recover the raw parameters from the enhanced Jump-Wing parameters we start by computing the raw b and ρ parameters.

The b parameter

Firstly, given the enhanced parameters we compute the total variance w_t using equation (2), i.e., $w_t = t \cdot v_t$. Then, from the sum of equations (4) and (5) we can recover the b parameter as:

$$b = \frac{\sqrt{w_t}}{2}\,(c_t + p_t)\ . \tag{8}$$

This equation is the same as the one given in reference [1]. Note that the b parameter can always be recovered from the Jump-Wing SVI parameters.

We now have two cases: when b equals zero and when b is different from zero. When b is zero, the Jump-Wing SVI parameterization is degenerate and the raw SVI parameters are not uniquely defined. However, looking at equation (1) we see that the only parameter that is really needed is a. Therefore we can set:

$$\begin{aligned} a &= w_t\,, \\ b &= 0\,, \\ \rho &= 0\,, \\ m &= 0\,, \\ \sigma &= 0\,. \end{aligned} \tag{9}$$

We continue the discussion assuming that b is different from zero.

The ρ parameter

Since $\sqrt{w_t}$ is always positive, we can multiply equations (4) and (5) by $\sqrt{w_t}$ and subtract the first from the second to obtain:

$$\sqrt{w_t}\cdot(c_t - p_t) = b\,(1+\rho-1+\rho) = 2\,b\,\rho\,,$$

so that we can recover the ρ parameter as:

$$\rho = \frac{c_t - p_t}{2\,b}\,\sqrt{w_t}\,. \tag{10}$$

Where we recall tha b is always different from zero so that equation (10) is always well defined. Note that this formula is different from the one given in reference [1], which assumes a perfect knowledge of the coefficients c_t and p_t. When the coefficients c_t and p_t are known with some error, equation (10) provides a better estimate of the ρ parameter, since we are averaging the two quantities with a similar error[1].

By combining together equations (8) and (10) we can express the ρ parameter solely in terms of the Jump-Wing SVI parameters:

$$\rho = \frac{c_t - p_t}{c_t + p_t}\,.$$

Since c_t and p_t are non negative, we have that $|\rho| \le 1$. Note that we cannot deduce the stronger condition $|\rho| < 1$.

5 The m parameter

The computation of the m parameter is more involved and requires a lengthier derivation. We split this derivation into a number of steps.

First equation for m and σ

Firstly we derive an expression for $m/\sqrt{m^2+\sigma^2}$. Rearranging the terms in equation (3) we have,

$$\frac{m}{\sqrt{m^2+\sigma^2}} = \rho - \frac{2\,\psi_t\,\sqrt{w_t}}{b}\,,$$

so that,

$$\frac{m}{\sqrt{m^2+\sigma^2}} = \beta\,, \tag{11}$$

[1] According to the traditional error propagation formula we reduce the error by a factor of $\sqrt{2}$.

where we have defined β as:

$$\beta = \rho - \frac{2\,\psi_t\,\sqrt{w_t}}{b}\,. \tag{12}$$

Note that this definition of β is the same as the one given in reference [1].

Assuming that σ is positive, from equation (11), we have that $\beta = 0$ implies $m = 0$. By extension, we define $m = 0$ when $\beta = 0$, regardless of the value of σ.

When β is different from zero, we can write the expression that links m with σ as:

$$\sqrt{m^2 + \sigma^2} = \frac{m}{\beta}\,, \tag{13}$$

this last equation will be used in the following computations.

Computing σ given m

Let us assume that both m and β are known and non zero. Then, since $\sqrt{m^2 + \sigma^2}$ is positive, from equation (13), we must have:

$$\text{sign}(\beta) = \text{sign}(m)\,.$$

We can now compute σ from equation (13) by observing that m=sign$(\beta)\sqrt{m^2}$, so that we have:

$$\sqrt{m^2 + \sigma^2} = \frac{\text{sign}(\beta)\sqrt{m^2}}{\beta}\,.$$

Since m is not zero we can write:

$$\frac{\text{sign}(\beta)}{\beta} = \frac{\sqrt{m^2 + \sigma^2}}{\sqrt{m^2}} = \sqrt{\frac{m^2 + \sigma^2}{m^2}} = \sqrt{1 + \frac{\sigma^2}{m^2}}\,.$$

By squaring both sides we obtain:

$$\frac{1}{\beta^2} = 1 + \frac{\sigma^2}{m^2}\,,$$

which implies:

$$\sigma^2 = \left(\frac{1}{\beta^2} - 1\right) \cdot m^2\,.$$

Recalling that we only accept non-negative values for σ, therefore by taking the square root of the above equation we obtain:

$$\sigma = \text{sign}(\beta) \cdot \sqrt{\frac{1}{\beta^2} - 1} \cdot m\,,$$

which can be written as:

$$\sigma = \alpha \cdot m\,, \tag{14}$$

by defining

$$\alpha = \text{sign}(\beta) \cdot \sqrt{\frac{1}{\beta^2} - 1}\,. \tag{15}$$

Note that also in this case the expression for α is the same as the one given in reference [1].

Second equation for m and σ

Equation (14) linearly relates m with σ, however we need a second equation to fully recover both parameters. In order to obtain this second equation we substitute $\sqrt{m^2 + \sigma^2}$ from equation (13) into equation (2), to obtain:

$$v_t \cdot t = a + b\,m\left(-\rho + \frac{1}{\beta}\right)\,, \tag{16}$$

we then considerz equation (6), i.e.,

$$\tilde{v}_t \cdot t = a + b\,\sigma\sqrt{1-\rho^2}\,.$$

Subtracting this expression from equation (16) we obtain:

$$(v_t - \tilde{v}_t)\cdot t = b\,m\left(-\rho + \frac{1}{\beta}\right) - b\,\sigma\sqrt{1-\rho^2}\,, \tag{17}$$

i.e. an equation that linearly relates m with σ, while the other parameters are known.

Final expression for m

We can now substitute σ from equation (14) into equation (17) to obtain:

$$(v_t - \tilde{v}_t)\cdot t = b\,m\left(-\rho + \frac{1}{\beta}\right) - b\,\alpha\,m\sqrt{1-\rho^2}\,,$$

which can be re-written as:

$$(v_t - \tilde{v}_t)\cdot t = b\,m\left[-\rho + \frac{1}{\beta} - \alpha\sqrt{1-\rho^2}\right]\,,$$

so that we have:

$$m = \frac{(v_t - \tilde{v}_t)\cdot t}{b\left[-\rho + \beta^{-1} - \alpha\sqrt{1-\rho^2}\right]}\,. \tag{18}$$

i.e. an explicit formula to recover the m parameter from the enhanced Jump-Wing SVI parameters and the previously computed b and ρ parameters.

Note that equation (18) is simpler, and therefore faster to compute, than the one given in reference [1].

6 Computation of the a and σ parameters

The a and σ parameters need to be computed together by distinguishing a few cases depending on the computed values of ρ and m.

Notice that, regardless of the values of ρ and m, with the help of equation (6), we can always write an expression for a as,

$$a = \tilde{v}_t \cdot t - b\,\sigma\sqrt{1-\rho^2}\,. \tag{19}$$

Then we distinguish two main cases: when m is zero and when m is different from zero.

Case when m is zero

When m equals zero equation (2) can be re-written as

$$v_t = \frac{a + b\,|\sigma|}{t} = \frac{a + b\,\sigma}{t}\,,$$

where the second equality is due to the fact that σ is not negative. Substituting a from equation (19) into the above expression we obtain:

$$v_t \cdot t = \tilde{v}_t \cdot t - b\,\sigma\sqrt{1-\rho^2} + b\,\sigma\,,$$

which becomes

$$(v_t - \tilde{v}_t)\cdot t = \sigma\cdot\left(1 - \sqrt{1-\rho^2}\right)\cdot b\,,$$

so that we have:

$$\sigma = \frac{(v_t - \tilde{v}_t)\cdot t}{\left(1 - \sqrt{1-\rho^2}\right)\cdot b}\,, \tag{20}$$

This expression can be used to compute the σ parameter only when $\rho \neq 0$.

When ρ is zero equation (20) cannot be used. In this case we can compute σ from b and ξ_t using equation (7), i.e.:

$$\sigma = \begin{cases} \dfrac{b}{\xi_t} & \text{if } \xi_t \neq 0\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{21}$$

This expression can be used to compute the σ parameter only when both $\rho = 0$ and $m = 0$. Note that equation (21) has not been given in reference [1].

Case when m is not zero

As seen in reference [1], the case when m is different from zero we simply use equation (14):

$$\sigma = \alpha \cdot m\,.$$

Finally, once the parameter σ is known, the a parameter can be recovered from equation (19), i.e.:

$$a = \tilde{v}_t \cdot t - b\sigma\sqrt{1-\rho^2}\,.$$

7 Vector computation of raw SVI parameters

Consider now a vector of enhanced Jump-Wing SVI parameters $\chi^i_{\text{ejw}} = \{v^i, \psi^i, p^i, c^i, \tilde{v}^i, \xi^i\}$, we want to compute the corresponding vector of raw SVI parameters $\chi^i_{\text{raw}} = \{a^i, b^i, \rho^i, m^i, \sigma^i\}$. We consider all indexes $i \in \Omega$, where Ω is an index set. Also, since we are interested in numerical computations, we define the error scale ε as a small positive number, e.g., $\varepsilon = 10^{-8}$.

Given the time to expiration t, we start by computing the vector w^i of total variance from the Jump-Wing SVI parameters using equation (2), i.e.:

$$w^i = v^i \cdot t \qquad \text{for all } i \in \Omega\,. \tag{22}$$

Then for all indexes $i \in \Omega$ we set the fallback values to the set the raw parameters as in equation (9), i.e.:

$$\begin{aligned} a^i &= w^i\,, \\ b^i &= 0\,, \\ \rho^i &= 0\,, \\ m^i &= 0\,, \\ \sigma^i &= 0\,, \end{aligned} \qquad \text{for all } i \in \Omega\,. \tag{23}$$

This is because all corner cases that cannot be handled by the following formulas will be set to the fallback values.

Computation of the b, ρ and m vectors

We compute the vector b^i using equation (8), i.e.:

$$b^i = \frac{c^i + p^i}{2\sqrt{w^i}} \qquad \text{for all } i \in \Omega\,. \tag{24}$$

We then define the set Ω' of indexes i such that b^i is numerically not zero[2], i.e.:

$$\Omega' = \left\{ i \in \Omega \quad \text{so that} \quad |b^i| \geq \varepsilon \right\}\,.$$

[2]Since the parameter is not dimensionless, we should in principle choose a scale for it, e.g., $\max_i \{b^i\}$, however this scale in practice is of the order of 1.

We now consider all indexes $i \in \Omega'$ and compute the ρ^i parameter using equation (10), i.e.:

$$\rho^i = \frac{c^i - p^i}{2\, b^i} \sqrt{w^i} \qquad \text{for all } i \in \Omega' \,. \tag{25}$$

Then we proceed with the computation of the m parameters. From equation (18) we have:

$$m^i = \frac{\left(v^i - \tilde{v}^i\right) \cdot t}{b^i \left[-\rho^i + \gamma^i - \alpha^i \sqrt{1 - (\rho^i)^2}\right]} \qquad \text{for all } i \in \Omega' \,. \tag{26}$$

where we defined γ^i as the inverse of β^i from (12) and α^i as in equations and (15), i.e.:

$$\gamma^i = \left(\rho^i - \frac{2\,\psi^i \sqrt{w^i}}{b^i}\right)^{-1} \qquad \text{and} \qquad \alpha^i = \text{sign}(\gamma^i) \cdot \sqrt{(\gamma^i)^2 - 1} \,. \tag{27}$$

Computation of the σ and a vectors

In order to properly compute the σ parameter we need to distinguish a number of cases and use either of equations (14), (20) or (21). Specifically we define:

$$\sigma = \begin{cases} \alpha^i \cdot m^i & \text{when } m^i \neq 0 \,, \\ \dfrac{\left(v^i - \tilde{v}^i\right) \cdot t}{\left[1 - \sqrt{1 - (\rho^i)^2}\right] \cdot b^i} & \text{when } m^i = 0 \text{ and } \rho^i \neq 0 \,, \\ \dfrac{b^i}{\xi^i} & \text{when } m^i = 0 \text{ and } \rho^i = 0 \text{ and } \xi^i \neq 0 \,. \end{cases} \tag{28}$$

The case not covered here are covered by the fallback values in equation (23).

Finally we compute the a parameter using equation (19), i.e.:

$$a^i = \tilde{v}^i \cdot t - b^i \, \sigma^i \sqrt{1 - (\rho^i)^2} \qquad \text{for all } i \in \Omega' \tag{29}$$

and the case when $b = 0$ is covered by the fallback values in equation (23).

8 Conclusions

We have shown how to compute the raw SVI parameters from the enhanced Jump-Wing SVI parameters in a vectorized way. This computation is useful for the interpolation and extrapolation of the SVI calibrations. While these computations are not very different from the ones given in reference [1], they are more explicit and easier to compute. Furthermore we have shown how to handle all the degenerate cases that can arise in practice.

References

[1] J. Gatheral, A. Jacquier *Arbitrage-free SVI volatility surfaces*, Quantitative Finance, https://arxiv.org/abs/1204.0646, 2014

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