TALOS

ANALYSIS

Futures Projection Rates and Basis Numbers

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Abstract

This paper explores the pricing variables of perpetual and futures contracts within the Talos portfolio management system. For perpetual contracts, we define the contract basis as the relative difference between the contract quote and the index price, introducing a 1-week equivalent rate to simplify basis interpretation. For futures contracts, we introduce the projection rates, defined as the continuously compounded rates used to project the index quote to other maturities. We provide three definitions of projection rates: the forward-starting rate, the zero-maturity rate, and the spot-index rate, each suited for different application contexts. In order to compute the forward prices with the forward-starting projection rates and the zero-maturity projection rates, one also needs to employ the corresponding basis numbers. Similarly, the spot-index projection rates are useful in purely continuous-time models, such as the popular Black-Scholes-Merton model often used to compute option prices. We illustrate these concepts with examples from market data on the Deribit exchange.

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1 Introduction

The digital asset market has grown notably with the introduction of various financial tools such as perpetual and futures contracts. These tools present traders with a range of strategies to interact with the digital financial markets, each having its own unique features and effects. Perpetual contracts, which do not have expiry dates, allow traders to hold positions as long as they have the necessary collateral. On the other side futures contracts, with set expiry dates, enable traders to take positions on asset prices at precise future times. It is crucial to understand these tools for effective risk management and strategy improvement in digital asset trading. This paper discusses the fundamental aspects of perpetual and futures contracts, explores the pricing variables associated with these contracts, and introduces the concept of projection rates to estimate the forward prices of digital assets.

In this dynamic environment the portfolio management system by Talos becomes an indispensable tool, equipping traders and investors with the knowledge and analytical capabilities to effectively manage and capitalize on the intricacies of linear derivatives on digital assets.

While the term *linear derivatives* often refers to instruments with linear or inverse payoffs, in this context, we use the term more broadly to describe financial tools with linear or quasi-linear payoffs; see reference [5] for a more detailed discussion of the related pricing functions. The Talos portfolio management system is designed to handle derivatives with both linear and inverse payoffs, ensuring comprehensive support for the diverse strategies employed in digital asset trading.

1.1 Market data

At Talos we believe that market data should be the center of any modern financial software, therefore we built the portfolio management system as a data-driven system.

Digital asset exchanges offer a variety of financial instruments for trading asset pairs like BTC/USD. These include:

- Spot markets, facilitating the immediate trading of a digital assets.
- Perpetual contracts, enabling leveraged positions not tied to any specific maturity.
- A range of futures contracts, allowing traders to take positions at different future maturities.

These instruments provide traders with the flexibility to align their trading strategies according to their market outlook, risk tolerance, and investment horizon. The diversity in instrument types and maturity dates also enriches the market ecosystem, fostering a dynamic and comprehensive trading environment.

Note that not all exchanges offer all these asset types. For example, there are exchanges that only offer futures and perpetual contracts and do not have spot facilities. On the other hand some exchanges, such as DeFi platforms, do not significantly cover margin trading contracts. As shown in reference [5], this is important because trading the same asset on different exchanges may offer some arbitrage opportunities. The below definitions of basis and projection rates are somewhat measuring these arbitrage opportunities.

Understanding the variety and characteristics of these instruments requires reliable market data. At Talos, we collect data from a multitude of providers to create data series for the portfolio management system. Accurate market data is crucial for reliable analyses and decision-making in financial undertakings. To ensure the integrity and accuracy of the data, Talos has implemented robust data-quality checks, as a consequence the data utilized is of high quality. The data in table 1 illustrates the quotes of perpetual and futures contracts for the BTC/USD pair on the Deribit exchange. This data is used to compute the basis numbers and projection rates discussed in this paper.

Contract Type	Name	Maturity	Index Price	Mark Price
PERPETUAL	BTC-PERPETUAL	*–None–*	66,938.67	66,955.93
FUTURE	BTC-25OCT24	2024-10-25	66,939.20	67,016.63
FUTURE	BTC-1NOV24	2024-11-01	66,938.15	67,144.75
FUTURE	BTC-29NOV24	2024-11-29	66,939.20	67,671.62
FUTURE	BTC-27DEC24	2024-12-27	66,962.23	68,208.20
FUTURE	BTC-28MAR25	2025-03-28	66,939.20	69,701.68
FUTURE	BTC-27JUN25	2025-06-27	66,939.20	71,189.02
FUTURE	BTC-26SEP25	2025-09-26	66,939.20	72,679.60

Table 1: Deribit market quotes for the Bitcoin/USD pair on 22 October 2024 at 13:00 UTC. Recall that all futures maturities are at 8:00 UTC.

2 Perpetual Contracts

Perpetual contracts represent a significant innovation in the domain of financial derivatives, especially within digital asset markets. Unlike traditional futures contracts, which have a predetermined expiration date, perpetual contracts do not possess an expiry date, allowing positions to be held indefinitely as long as the necessary collateral is maintained.

The trading dynamics of perpetual contracts also differ from those of futures contracts, especially regarding the settlement process, which occurs continuously in the case of perpetual contracts as opposed to at a set date in the case of traditional futures contracts.

2.1 The perpetual contracts basis

Given a digital asset its spot price represents the current market price at which the asset can be bought for immediate delivery (or sold depending if it is a bid or an ask price). The quoted price reflects the immediate equilibrium between supply and demand for that asset. In digital-asset exchanges the spot price serves as the most straightforward representation of an asset's current value and is often considered a reference price for the associated derivatives. On the other hand a perpetual-contract quote refers to the price at which traders can enter a contract to buy or sell a digital asset at a later time, without a predetermined expiration date. Perpetual contract quotes can deviate from the current market price of the underlying asset due to factors such as funding rates, leverage, and market sentiment. The perpetual contract quote is affected by traders' expectations of the asset's future price movements and the dynamics of the perpetual market itself.

It should be noted that perpetual contracts are not settled with respect to the spot price, instead they are settled with respect to an index price based on the underlying asset value. The index price is often a weighted average of the spot prices of the asset on a number of markets and is quoted by the exchange where the perpetual contract is traded.

We denote with I_t the index quote at the time t and with F_t the perpetual contract quote at the same time. The basis of a perpetual contract measures the discrepancy between the contract quote and the index price. Specifically, we define the *contract basis* B_t^c implicitly from this expression:

$$F_t = I_t \cdot (1 + B_t^c) \ . \tag{1}$$

Here the superscript ^{c} reminds us that we are dealing with a contract and the subscript $_t$ is a reminder that the basis is time-dependent. More explicitly, consistently with definitions (1), the basis B_t^c is defined as

$$B_t^c = \frac{F_t}{I_t} - 1 \, . \tag{2}$$

Since both F_t and I_t are positive, the multiplicative basis B_t^c can assume any value greater than -1. When the contract basis is negative, the perpetual is quoted at a discount relative to the index. Conversely, when the basis is positive, the contract is quoted at a premium relative to the index price. Equilibrium between the perpetual quote and its index is achieved when the basis is precisely zero.

Contract Type	Name	Contr. Basis	Weekly Rate
PERPETUAL	BTC-PERPETUAL	2.58	1.34
FUTURE	BTC-25OCT24	11.57	6.03
FUTURE	BTC-1NOV24	30.71	16.01
FUTURE	BTC-29NOV24	109.42	57.05
FUTURE	BTC-27DEC24	189.58	98.85
FUTURE	BTC-28MAR25	412.68	215.19
FUTURE	BTC-27JUN25	634.88	331.04
FUTURE	BTC-26SEP25	857.55	447.15

Table 2: Contract basis numbers and weekly equivalent rates for all contracts listed in table 1. The basis numbers are expressed in basis points (bips) and the weekly rates are expressed in percentage points (%). Note that practitioners are usually interested only in the perpetual basis and the basis of the nearest futures contract.

The first row in table 2 shows the computation of the contract basis for the derivatives listed in table 1.

In order to make the interpretation of the basis numbers more straightforward, so that they can be compared with the projection rates defined below, we define the concept of the 1-week equivalent rate. For any given multiplicative basis number, we define the *1-week equivalent rate* as the constant rate that would be necessary to achieve the same percentage variation of the basis in one week.

Talos publishes periodically the multiplicative contract basis numbers for all available perpetual contracts.

3 Forward Starting Projection Rates

Futures contracts are standardized agreements between two parties to buy or sell an asset at a predetermined price at a specified future date and time. Unlike spot trading, where the transaction and asset delivery occur immediately, futures contracts entail a commitment to complete the transaction at a later date. This mechanism allows traders and investors to hedge and speculate against price movements without the immediate need for capital outlay or asset transfer.

Comparatively, perpetual contracts, as discussed in section 2, allow for similar speculative and hedging activities but without a predetermined expiration date. This key difference means that traders can hold their positions indefinitely in a perpetual contract, while in a futures contract, positions are settled at the contract's expiration date. See reference [5] for a comprehensive discussion on the pricing function of perpetual and futures contracts.

Arbitrage relationships play a pivotal role in deciphering the pricing and dynamics of financial instruments in the market. For instance, reference [3] elucidates a simple arbitrage-free relationship between the spot price and the forward price. Although theoretically sound, the derivation of this relationship often encounters deviations in digital asset markets, where arbitrage opportunities may arise. Therefore, we aim to provide a definition of projection rates that stands independently from any arbitrage relationship.

3.1 Continuous-trading related nuances

Differently from traditional equity markets, digital asset markets are characterized by their 24/7 trading hours, which allow for continuous trading of perpetual and futures contracts. This continuous trading environment provides a unique opportunity to observe the dynamics of the market, however, it also introduces challenges in the computation of projection rates.

Threshold maturity hours

Consider a specific trading pair, for instance Bitcoin/USD, and all the futures contracts associated with this pair on a given exchange like, e.g., the Deribit exchange. We focus on the futures contract with the shortest maturity. It's a common practice for traders to rollover their contracts to the subsequent maturity (i.e., the second shortest maturity) a few hours before expiration to circumvent delivery. We observed that quotes associated with maturities shorter than approximately 3 hours, tend to lose liquidity and eventually are removed from the list of actively traded futures. We introduce the term *threshold-maturity-hours* to denote the minimum number of hours required in the futures for a quote to be deemed valid. In the aforementioned scenario the threshold maturity hours, denoted with T_{min}, is 3 hours. Talos periodically reviews the threshold maturity hours to ensure that the quotes utilized to bootstrap the projection rates always exhibit adequate liquidity.

Date/time conventions

As illustrated in Table 1, at any given moment and for the same specified trading pair, a variety of futures contracts with different maturities are available. Unlike traditional commodity futures, digital asset futures are traded 24 hours a day, 7 days a week. Therefore, when calculating the residual maturity, it is crucial to consider not only the number of days before expiry but also the exact time.

When the need arises to compute the elapsed period between two distinct events, we use the difference in the number of days, adding a fraction so that an hour is counted as 1/24 days, a minute is counted as 1/60 hours, and so on. The fractional number of days obtained in this manner is then divided by the factor 365, to obtain the year fraction between the two events in time.

3.2 Forward-starting projection rates

Suppose we have a collection of n futures contracts listed on a specific exchange, all based on the same underlying pair, with the same settlement currency and the same payoff type (i.e. liner or inverse). We arrange these contracts in ascending order of their time to maturity, with corresponding quotes marked F_1, F_2, ..., F_n, and time-to-maturity in year fractions at $T_1 < T_2 < \ldots < T_n$. Note that we restrict our selection to contracts expiring after the minimum threshold hours, i.e. with $T_1 \geq T_{\mathrm{min}}$. For simplicity we denote the current time as t: this is the reference time at which we observe the market data. We call T_1 the nearest futures maturity and F_1 the nearest futures quote. Similarly T_n is the farthest futures maturity and F_n is the farthest futures quote.

With all these definitions in place, we can now formally define the *forward-starting projection rates* r_j's at the maturity nodes as,

$$r_j = \frac{1}{T_j - T_1} \log\left(\frac{F_j}{F_1}\right) \qquad \text{for} \quad j = 2, \ldots, n \,. \tag{3}$$

Given the nearest futures quote F_1 we can invert equation (3) to obtain the futures quotes in terms of the forward-starting projection rates:

$$F_j = F_1 \, e^{r_j \left(T_j - T_1\right)} \qquad \text{for} \quad j = 2, \ldots, n \,. \tag{4}$$

If we were to assume these projection rates to define some kind of zero rates between T_1 and T_j then they would reflect the market expectation for projecting F_1 from T_1 to T_j, for $j = 2, \ldots, n$ (which provides a justification for the *forward-starting projection rate* nomenclature).

In the second column of table 3, we show the forward-starting projection rates r_j's computed using the contract *BTC-25OCT24* as the contract expiring at T_1, for the dataset of table 1. Similarly in figure 1 we show a plot of the projection rates computed using the market data of table 3. As evident from both the table and the figure, at the time we captured the market data, the

j	Maturity	r_j (%)	f_j (%)	z_j (%)	s_j (%)
1	BTC-25OCT24	9.96	9.96	9.96	15.11
2	BTC-1NOV24	9.96	9.96	9.96	11.43
3	BTC-29NOV24	10.14	10.19	10.13	10.51
4	BTC-27DEC24	10.21	10.30	10.20	10.42
5	BTC-28MAR25	9.31	8.69	9.32	9.41
6	BTC-27JUN25	9.00	8.47	9.01	9.07
7	BTC-26SEP25	8.81	8.31	8.82	8.86

Table 3: Node values, for $j = 1, \ldots, 7$, of the forward-starting projection rates r_j's, the forward rates f_j's, the zero-maturity projection rates z_j's, and spot-index projection rates s_j's, computed using the futures quotes listed in table 1.

forward prices shows a typical upward sloping curve of a *contango market*. As a consequence the corresponding projection rates are all positive.

3.3 Interpretation of the projection rate

Bitcoin is often referred to as *digital gold* due to its scarcity and store-of-value characteristics, akin to physical gold. For example, Ferdinando Ametrano, a notable figure in the cryptocurrency and blockchain space, has explored this comparison extensively in his work (see, e.g., reference [1]). While Bitcoin does not pay dividends, it can be considered to have a convenience yield. Convenience yields, typically associated with commodities, represent the non-monetary benefits of holding an asset rather than a derivative contract on the asset itself.

In the case of Bitcoin, the convenience yield may be interpreted as the benefits derived from holding a decentralized, borderless, and censorship-resistant form of money, offering a hedge against inflation and monetary policy changes. Additionally, there is a security element to consider: holding Bitcoin directly in cold storage eliminates counterparty risks, such as those arising from centralized exchanges or DeFi smart contracts. When evaluating the opportunity cost of using Bitcoin to collateralize a derivatives position, traders may weigh the foregone yield against the expected risk of capital loss, which could be thought of as a form of implied credit spread rather than a traditional convenience yield[1].

For other digital assets the convenience may come from different sources. For instance, the Ether token's convenience could arise from its utility as gas for running smart contracts on the Ethereum blockchain. In this work we assume that any digital asset has a certain convenience in holding it directly as opposed to entering into a contract for its future delivery. This convenience, encompassing both intrinsic benefits and security considerations, is reflected in a constant yield q.

In the conventional Black-Scholes option-pricing framework for a commodity with a continuously-paying convenience yield q in a market with a continuously compounded interest rate i, the forward-forward arbitrage relationship is given by:

$$F_b = F_a \cdot e^{(i-q)(T_b-T_a)} \quad \text{with} \quad T_b > T_a \,,$$

where F_a and F_b denote the quotes of two forward contracts, one maturing at T_a and the other at T_b. The term $i - q$ in the exponent signifies that the exponential growth is proportional to the difference between the interest rate i and the convenience yield q. Comparing this equation with equation (4) we observe that the projection rate can be thought as the difference between the risk-free rate and the convenience yield:

$$r = i - q \,.$$

[1]We thank Kyle Downey for this insightful suggestion regarding the interplay between convenience yield and implied credit spreads in the context of digital asset holdings.

Given the observations in reference [4] on the ambiguity of selecting appropriate interest rates for digital assets, and the undefined nature of a convenience yield for digital tokens, we calculate the projection rate as provided, without additional assumptions, and base the forward price computation directly on r.

3.4 Interpolation and extrapolation of projection rates

We previously defined the forward-starting projection rates at the maturities of listed futures contracts. A natural question arises: how can we extend this definition to cover maturities that are not listed on the market?

Several scenarios illustrate the need for projection rates beyond listed futures maturities, such as:

1. Estimating the forward price when its maturity does not match any listed futures contract.
2. Establishing a time-invariant term structure of constant-maturity projection rates, such as computing daily projection rates for fixed tenors like 1-week, 1-month, and 3-month intervals.

To address this need, we must find an effective method to interpolate between the maturity dates of futures contracts. Additionally, we need a reliable approach for extrapolating projection rates for maturities that either precede T_2 or extend beyond T_n.

In general, given a maturity T, we would like to write the expected forward price as a function of the nearest-futures projection rate function $r(T)$, so that

$$F(T) = F_1 \cdot e^{r(T)(T-T_1)}\,, \tag{5}$$

with $r(T)$ is defined for all $T > t$.

Extrapolation of projection rates

Let's start with the extrapolation of the forward-starting projection rates. We assume a constant flat extrapolation both before and after the observed rates. Therefore when extrapolating to dates that are earlier than the second-nearest futures maturity, we define:

$$r(T) = r_2 \qquad \text{for} \quad t \le T \le T_1\,.$$

In particular we define

$$r_1 = r(T_1) = r_2\,.$$

Similarly, when extrapolating to dates that are later than the farthest futures-contract maturity, we define:

$$r(T) = r_n \qquad \text{for} \quad T \ge T_n\,,$$

where we recall that T_n is the maturity of the longest-dated futures contract.

Interpolation of projection rates

The interpolation of the projection rates is a bit trickier since we would like to use it in a large number of quantitative models, such as the Black-Scholes-Merton model for pricing derivative assets (see the later discussion in this paper about this topic). On the other hand we do not want to use exceedingly complicated interpolation methods, such as those based on cubic splines (see, e.g., reference [2]).

Taking inspiration from the interest-rate world, we interpolate, in each maturity segment, linearly on the logarithm of the forward price. As shown in reference [2] this method corresponds to

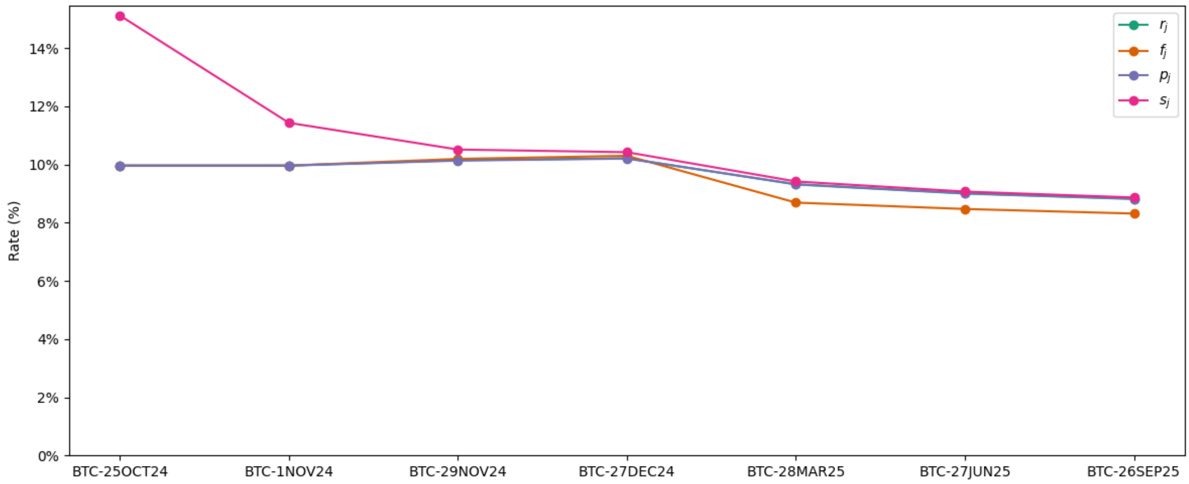

Figure 1: Plot of the node forward-starting projection rates r_j's, the forward rates f_j's, the zero-maturity projection rates z_j's, and spot-index projection rates s_j's, presented in in table 3.

piecewise-constant instantaneous forward curves. In practice given a generic maturity T that is strictly included in the j-segment, i.e. so that we have

$$T_{j-1} < T < T_j \qquad \text{for some} \;\; j = 2, \ldots, n\,,$$

we define the segment forward rate f_j so that:

$$F(T) = F_{j-1} \cdot e^{f_j\dot{(}T - T_{j-1})}\,. \tag{6}$$

For consistency, $F(T)$ must be such that $F(T_j) = F_j$, i.e. the forward price at T_j must be the futures price F_j:

$$F_j = F_{j-1} \cdot e^{f_j\dot{(}T_j - T_{j-1})}\,.$$

By inverting this expression we can derive an equation for the segment forward rates f_j:

$$f_j = \frac{1}{T_j - T_{j-1}} \log\left(\frac{F_j}{F_{j-1}}\right) \qquad \text{for all} \;\; j = 2, \ldots, n\,. \tag{7}$$

Note that $f_2 = r_2$. We also define the forward rate f_1 as the projection rate r_1, i.e. $f_1 = r_1$ and also note that we have $f_1 = f_2$.

Table 3 provides the values for the forward rates f_j's for the market data of table 1.

Forward-starting projection-rate function

We now have all the necessary ingredients to define the forward-starting projection rate function $r(T)$. We can take $F(T)$ from expression (5) and substitute it into equation (6) to obtain:

$$F_1\, e^{r(T)(T-T_1)} = F_{j-1} \cdot e^{f_j\dot{(}T - T_j)}\,.$$

We divide both sides by F_1 and take the natural logarithm of both sides to obtain:

$$r(T)(T - T_1) = \log\left(\frac{F_{j-1}}{F_1}\right) + f_j\dot{(}T - T_j)\,,$$

where we used the property of the logarithm $\log(a \cdot b) = \log(a) + \log(b)$. We can further simplify this expression by using equation (3) for $j - 1$ to give:

$$r(T)(T - T_1) = r_{j-1}\left(T_{j-1} - T_1\right) + f_j\dot{(}T - T_j)\,,$$

Tenor	Maturity	Forw. Price	r_j (%)	s_j (%)	z_j (%)
1D	2024-10-23	66,983.88	9.96	24.35	9.96
2D	2024-10-24	67,002.16	9.96	17.16	9.96
3D	2024-10-25	67,020.44	9.96	14.76	9.96
1W	2024-10-29	67,093.63	9.96	12.02	9.96
2W	2024-11-05	67,223.67	10.05	11.06	10.03
3W	2024-11-12	67,355.16	10.10	10.77	10.08
1M	2024-11-21	67,524.59	10.13	10.59	10.11
2M	2024-12-21	68,096.86	10.20	10.43	10.19
3M	2025-01-21	68,618.68	9.78	9.94	9.78
6M	2025-04-22	70,110.56	9.19	9.28	9.20
9M	2025-07-22	71,598.84	8.93	9.00	8.94
1Y	2025-10-22	73,140.93	8.81	8.86	8.82

Table 4: Tenor values of the node forward-starting projection rates r_j's, the forward rates f_j's, the zero-maturity projection rates z_j's, and spot-index projection rates s_j's, for maturities dates computed for tenors of 1-day, 2-day, 3-day, 1-week, 2-weeks, 3-weeks, 1-month, 2-months, 3-months, 6-months, 9-months, and 1-year.

which finally yields

$$r(T) = r_{j-1} \cdot \frac{T_{j-1} - T_1}{T - T_1} + f_j \cdot \frac{T - T_j}{T - T_1} \qquad \text{for} \quad T_{j-1} \leq T \leq T_j \,. \tag{8}$$

This equation provides the time-dependent forward-starting projection-rate function $r(T)$ in terms of the projection rates r_j's and the forward rates f_j's.

With the above definitions for the extrapolation of $F(T)$ and equation (6) for the interpolation of the forward prices between nodes, we can compute the expected forward price at any future date. For example in table 4 we lists the forward prices and all projection rates for maturities dates computed for tenors of 1-day, 2-day, 3-day, 1-week, 2-weeks, 3-weeks, 1-month, 2-months, 3-months, 6-months, 9-months, and 1-year. In figure 2 we plot the forward prices listed in table 4. Similarly in figure 3 we plot the projection rates $r(T)$'s for the same maturities.

When computing the (risk-neutral) values of derivatives at a specific date, we can use the projection rates to determine the expected forward prices. However, since the time to the maturity T_1 changes at all instants in time, we cannot use the forward-starting projection rates at a different date/time from the one at which they were computed. In order to overcome this limitation, we introduce the concept of zero-maturity projection rates in the next section.

4 Zero Maturity Basis and Zero Maturity Projection Rates

In section 2, we elaborate on the concept of basis for perpetual contracts. In this section, we expand on the same concept for futures contracts. The discussion for futures contract basis is important for scenarios such as:

1. Employing futures in a strategy alongside the spot price.
2. Incorporating futures in a strategy with a perpetual contracts.
3. In the absence of perpetual contracts, substituting a perpetual contract with the shortest-maturity futures.

We first introduce the concept of nearest-futures basis, which is the basis defined using the next-expiry contract. Then we define the zero-maturity futures quote, which is the *projection* of the first-maturity quote to the current time.

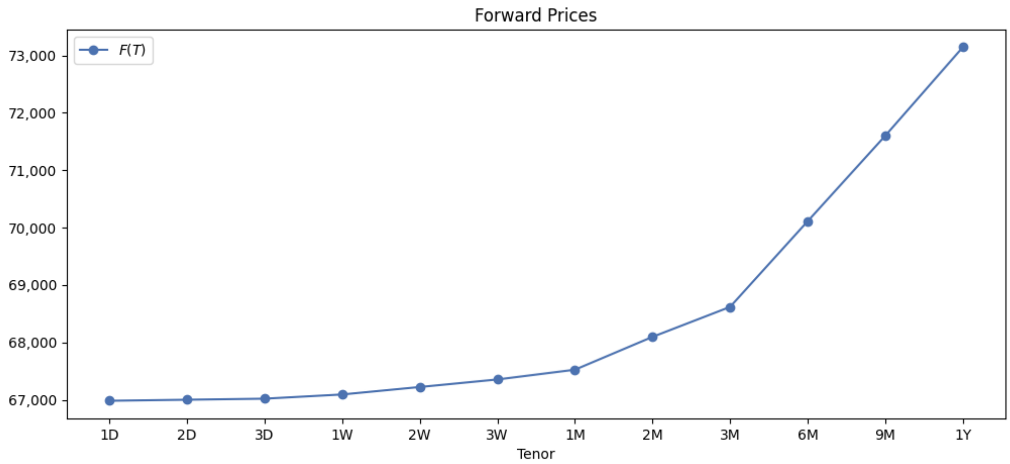

Figure 2: Plot of the forward prices computed using the data of table 4. Note that the tenor spacings are not drawn to scale of the actual time between nodes.

4.1 Nearest futures basis

We notice in the market data of table 1 that the index associated with futures contracts on the same underlying asset is *not* the same for all futures contracts. This is due the technicalities of the index computation, which may involve a weighted average of spot prices on different exchanges. Therefore, at any given time t, we define the median index price I_t as the median of the index prices of all futures contracts. For example in table 1 the median index price is $I_t = 66,939.20$.

Then we define the *nearest-futures multiplicative basis* as

$$B_t^{\mathrm{f1}} = \frac{F_1}{I_t} - 1\,. \tag{9}$$

In the symbol of basis the superscript $^{\mathrm{f1}}$ reminds us that the basis is defined using F_1 as reference quote.

By inverting equation (9), we can express the first-maturity futures quote in terms of the corresponding basis:

$$F_1 = I_t \cdot \left(1 + B_t^{\mathrm{f1}}\right)\,. \tag{10}$$

We substitute F_1 from this expression into equation (5) to obtain the projection rate function in terms of the nearest-futures basis and $F(T)$:

$$F(T) = I_t \cdot \left(1 + B_t^{\mathrm{f1}}\right) \cdot e^{p(T)(T-T_1)}\,. \tag{11}$$

Note that it is not possible to compare different values of the nearest-futures basis at different dates, because the time to next expiry T_1 is not constant. We solve this problem in the next subsection by creating the zero-maturity futures basis.

4.2 The zero-maturity projection rate

In an attempt to remove the dependence from varying time-to-next-expiry T_1, we project the first-maturity quote F_1 to the current time t. We can think the current time t as the virtual maturity time of the zero node, i.e. we can set $T_0 = t$. Therefore we define the zero-maturity futures virtual quote F_0 as,

$$F_0 = F_1\, e^{-r_1(T_1-t)}\,, \tag{12}$$

where we recall that, by definition, $r_1 = r_2$. We then invert this equation to find an expression for F_1 in terms of F_0:

$$F_1 = F_0 \, e^{r_1(T_1 - T_0)} \, . \tag{13}$$

This equation states that the first-maturity futures quote F_1 is the projection of the zero-maturity futures quote F_0 from the zero-node T_0 to the first node T_1 using the forward-starting projection rate r_1.

At this point we substitute F_1 from expression (13) into equation (5). In this way we obtain:

$$F(T) = F_0 \, e^{r_1(T_1 - t) + r(T)(T - T_1)} \, ,$$

so that

$$F(T) = F_0 \, e^{z(T)(T-t)} \, , \tag{14}$$

where the *zero-maturity projection-rate function* $z(T)$ is defined as

$$z(T) = \frac{r_1 \, (T_1 - t) + r(T)(T - T_1)}{T - t} \, . \tag{15}$$

Note that, in going from $r(T)$ to $z(T)$, we have removed the dependence from the forward-starting projection rate r_1, in this way we can compare $z(T)$ at different dates for any given constant maturity T.

We compute the zero-maturity projection rate at the node dates T_j, where we have $r(T_j) = r_j$, i.e.:

$$z(T_j) = r_1 \cdot \frac{T_1 - t}{T_j - t} + r_j \cdot \frac{T_j - T_1}{T_j - t} \, , \tag{16}$$

which simplifies to $z(T_1) = r_1$ for $j = 1$. For $j > 1$ equation (16) states that the zero-maturity projection rate at the node dates is equal to the weighted average of r_1 and r_j with weights proportional to the *distance* of T_j from t.

4.3 The zero-maturity futures basis

The value of F_0 defined in equation (12) can be viewed as the unbiased quote of an hypothetical futures contract that is about to expire immediately, i.e. at time T_0. In a perfectly efficient market with no arbitrage, the value of F_0 should be close to the spot index quote I_t. I.e. we expect to have

$$F_0 \simeq I_t \, .$$

In reality this isn't always the case and there might exist persistent differences between F_0 and I_t. In order to measure this difference we define the *zero-maturity futures basis* B^z_t is as:

$$B^{\mathrm{z}}_{\mathrm{t}} = \frac{F_0}{I_t} - 1 \, . \tag{17}$$

We invert this expression to obtain the the zero-maturity futures quote F_0 in terms of the median index price I_t and the zero-maturity futures basis:

$$F_0 = I_t \cdot (1 + B^z_t) \, . \tag{18}$$

We substitute F_0 from this expression into equation (14) to obtain

$$\boxed{F(T) = I_t \cdot (1 + B^z_t) \cdot e^{z(T)(T-t)} \, . \qquad (19)}$$

This equation is the major result of this paper and provides a functional form for the projected forward price in terms of the zero-maturity projection rate and the zero-maturity futures basis. Note how both B^z and z can be either observed/computed directly from market data, or simulated in risk scenarios. However, while the zero-maturity projection rate $z(T)$ is a good candidate for an

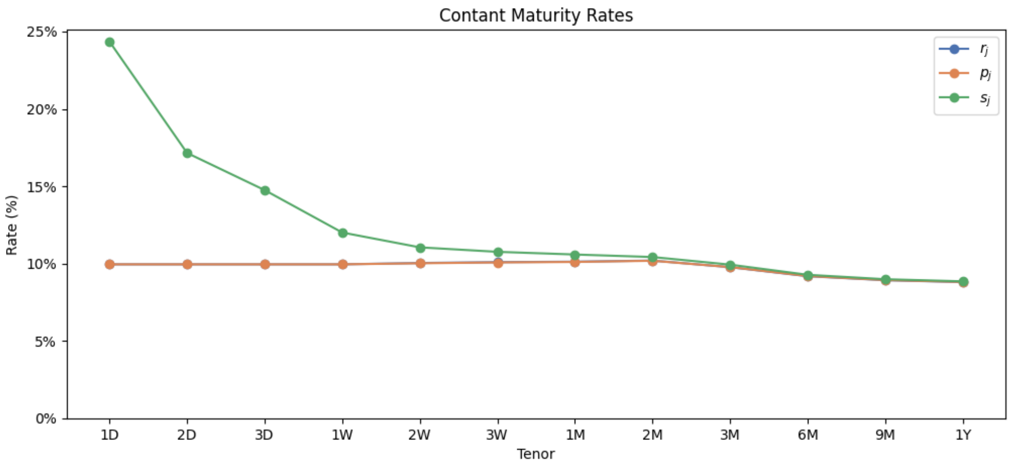

Figure 3: Term structure of interpolated projection rates listed in table 4 and plotted in figure 2. Note that the tenors spacing are not drawn to scale of the actual time in between.

invariant to be used in risk simulations, the zero-maturity futures basis B_t^z must be larger than -1. Hence, B_t^z is not an invariant, while $\log(1 + B_t^z)$ could be used as an invariant.

With the market data of table 1 we can compute the zero-maturity quote F_0 to be 66965.60 and the zero-maturity futures basis B_t^z to be 3.94 basis points.

Finally we remark that in order to compute the zero-maturity projection curve $z(T)$ at all maturities T, we need at least two futures quotes, i.e. F_1 and F_2. For consistency we also requires at least two observations of the index price.

5 Spot Index Projection Rates

Equation (19) provides an expression for the market expectation of the forward price $F(T)$ starting from the median index quote I_t, modified by the multiplicative zero-maturity futures basis, and capitalized by the zero-maturity projection rate. This expression should be used, for example, when evaluating a portfolio of futures contracts held both in long and short positions.

One thing to note in expression (19) is that the difference between the median index quote I_t and the zero-maturity futures F_0 is observed instantaneously at every time t. In the environment of stochastic models, this difference could be modeled as a jump process that happens instantaneously after time t. This jump process is not included in the zero-maturity projection rate $z(T)$, which could be derived by a continuous-time process. Hence we need to derive a definition of a projection rate that can be used in stochastic models that do not allow jumps. For example the Black-Scholes model, traditionally used in option pricing, does not allows for jumps at any point in time and uses a continuous-time process to model the evolution of the underlying asset.

In order to obtain a projection rate that can be used in stochastic models without jumps we introduce the *spot-index projection rate* $s(T)$ so that

$$F(T) = I_t \cdot e^{s(T)(T-t)} \,. \tag{20}$$

Substituting $F(T)$ from this expression into equation (19) we obtain:

$$I_t \cdot e^{s(T)(T-t)} = I_t \cdot (1 + B_t^z) \cdot e^{z(T)(T-t)} \,,$$

which can be simplified to yield:

$$\boxed{s(T) = \frac{\log\left(1 + B^z_t\right)}{T - t} + z(T)\,. \qquad (21)}$$

This equation is another important result of this paper and provides a relationship between the zero-maturity basis and the zero-maturity projection rates and the spot-index projection rates. Notice that if the zero-maturity rate $z(T)$ is an invariant, as mentioned earlier, since $\log(1 + B^z_t)$ is also an invariant so is the spot-index projection rate $s(t)$.

It is easy to see from equation (21) that when the zero-maturity futures basis B^z_t is zero we simply have:

$$s(T) = z(T)\,,$$

i.e., the spot-index projection rate equals the zero-maturity projection rate. Also, for asymptotically large T the two definitions of projection rates converge to the same value, i.e.:

$$\lim_{T\to\infty} \frac{s(T)}{z(T)} = 1\,.$$

We stress again how the spot-index projection rate $s(T)$ should be used, for example, in the Black-Scholes model when pricing options. Given the futures quotes F_j's and the median index price I_t we can compute the spot-index projection rates $s(T_j)$'s at the maturity dates T_j's:

$$s_j = s(T_j) = \frac{1}{T_j - t} \cdot \log\left(\frac{F_j}{I_t}\right) \qquad \text{for all } \; j = 1, \ldots, n\,. \qquad (22)$$

The values of the spot-index projection rate at the maturity nodes can computed from the market data of table 1 are reported in table 3 and plotted in figure 1. The spot-index projection rate at fixed tenor maturities is listed in table 4 and plotted in figure 3. It can be noticed from both the tables and the figures that for short maturities the second term in equation (21) tends to become large. This fact happens because for smaller and smaller maturities we are trying to *squeeze* the jump variation of the basis term into a rate proportional to $T - t$.

6 Conclusions

This study explored the financial variables associated with perpetual and futures contracts. The basis for perpetual contracts was examined by comparing the index price and perpetual contract quotes. The discussion then moved to futures contracts, introducing the concept of forward-starting projection rates to model price expectations across different maturity points. We complemented this rate by introducing the nearest-futures basis. We then defined an interpolation/extrapolation policy that allowed us to compute the forward-starting projection rates for fixed tenors, useful for risk simulations.

While the forward-starting projection rates are useful for pricing derivatives, they are not suitable for risk simulations. In order to define a basis and a curve that can also be used in risk simulations we introduced the zero-maturity futures basis an the zero-maturity projection rate. We then switched to the spot-index projection rate, which is a rate that can be used in stochastic models that do not allow jumps and we showed how it can be computed from the zero-maturity futures basis and the zero-maturity projection rate.

For all the rates and basis introduced we showed how they can be computed from the market data of table 1 and we provided the numerical values for the rates and basis in table 3.

In conclusion, the concepts introduced in this paper are crucial for the effective management of a portfolio of linear digital assets. Finally, the study was performed using the Talos portfolio management software, which provides a comprehensive set of tools for managing digital asset portfolios.

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